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Polynomial and Rational Matrices

Applications in Dynamical Systems Theory

Springer

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Preface

This monograph covers the selected applications of polynomial and rational matrices to the theory of both continuous-time and discrete-time linear systems. It is an extended English version of its preceding Polish edition, which was based on the lectures delivered by the author to the Ph.D. students of the Faculty of Electrical Engineering at Warsaw University of Technology during the academic year 2003/2004.

The monograph consists of eight chapters, an appendix and a list of references.

Chapter 1 is devoted to polynomial matrices. It covers the following topics: basic operations on polynomial matrices, the generalised Bézoute theorem, the Cayley–Hamilton theorem, elementary operations on polynomial matrices, the choosing of a basis for a space of polynomial matrices, equivalent polynomial matrices, reduced row matrices and reduced column matrices, the Smith canonical form of polynomial matrices, elementary divisors and zeros of polynomial matrices, similarity of polynomial matrices, the Frobenius and Jordan canonical forms, cyclic matrices, pairs of polynomial matrices, the greatest common divisors and the smallest common multiplicities of matrices, the generalised Bezoute identity, regular and singular matrix pencil decompositions, and the Weierstrass–Kronecker canonical form of a matrix pencil.

Rational functions and matrices are discussed in Chap. 2. With the basic definitions and operations on rational functions introduced at the beginning, the following issues are subsequently addressed: decomposition into the sum of rational functions, operations on rational matrices, the decomposition of a matrix into the sum of rational matrices, the inverse matrix of a polynomial matrix and its reducibility, the McMillan canonical form of rational matrices, the first factorization of rational matrices and the application of rational matrices in the synthesis of control systems.

Chapter 3 addresses normal matrices and systems. A rational matrix is called normal if every non-zero minor of size 2 of the polynomial matrix of the denominator is divisible by the minimal polynomial of this matrix. It has been proved that a rational matrix is normal if and only if its McMillan polynomial is equal to the smallest common denominator of all the elements of the rational matrix. Further, the following issues are discussed: the fractional forms of normal matrices, the sum and product of normal matrices, the inverse matrix of a normal matrix, the decomposition of normal matrices into the sum of normal matrices, the structural decomposition of normal matrices, the normalisation of matrices via feedback and electrical circuits as examples of normal systems.

The problem of the realisation of normal matrices is addressed in Chap. 4. The problem formulation is provided; further the following issues are discussed: necessary and sufficient conditions for the existence of minimal and cyclic realisations, methods of computing the realisation with the state matrix in both the Frobenius and Jordan canonical forms, structural stability and the computation of the normal transfer function matrix

Chapter 5 is devoted to normal singular systems. In particular it focuses on discrete singular systems, cyclic pairs of matrices, the normal inverse matrices of cyclic pairs, normal transfer matrices, reachability and cyclicity of singular systems, cyclicity of feedback systems, computation of equivalent standard systems for singular systems. It is shown that electrical circuits consisting of resistances and inductances or resistances and capacities, together with ideal voltage (current) sources, constitute examples of singular continuous-time systems. Both the Kalman decomposition and the structural decomposition of the transfer matrix are generalised to the case of singular systems.

Polynomial matrix equations, both rational and algebraic, are discussed in Chap. 6. The chapter begins with unilateral polynomial equations with two unknown matrices. Subsequently the following issues are addressed: the computation of minimal degree solutions to matrix equations, bilateral polynomial equations, the computation of rational solutions to polynomial equations, matrix equations of the m-th order, the Kronecker product of matrices and its applications, and the methods for computing solutions to Sylvester and Lapunov matrix equations.

Chapter 7, the last one, is devoted to the problem of realisation and perfect observers for linear systems. A new method for computing minimal realisation for a given improper transfer matrix is provided together with the existence conditions; subsequently the methods for computing full and reduced order observers, as well as functional perfect observers, for 1D and 2D systems are given.

In Chap. 8 some new results (published and unpublished) are presented on positive linear discrete-time and continuous-time systems with delays: asymptotic and robust stability, reachability, minimum energy control and positive realisation problem.

The Appendix contains some basic definitions and theorems pertaining to the controllability and observability of linear systems.

The monograph contains some original results of the author, most of which have already been published.

It is haped that this monograph will be of value to Ph.D. students and researchers from the field of control theory and circuit theory. It can be also recommended for undergraduates in electrical engineering, electronics, mechatronics and computer engineering.

I would like to express my gratitude to Professors M. Busłowicz and J. Klamka, the reviewers of the Polish version of the book, for their valuable comments and

suggestions, which helped to improve this monograph. I also wish to thank my Ph.D. students, the first readers of the manuscript, for their remarks.

I wish to extend my special thanks to my Ph.D. students Maciej Twardy, Konrad Markowski and Stefan Krzemiński for their valuable help in the preparation of this English edition.

T. Kaczorek

Contents

Notation

$\mathbf{A}$	matrix
$\mathbf{A}^T$	transpose of $\mathbf{A}$
$\mathbf{A}^*$	conjugate of $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$
Adj $\mathbf{A}$	adjoint (adjugate) of $\mathbf{A}$
$\mathbf{A}(s)$	polynomial matrix
$\mathbf{A}_J$	Jordan canonical form of $\mathbf{A}$
$\mathbf{A}_S(s)$	Smith canonical form of $\mathbf{A}(s)$
det $\mathbf{A}$	determinant of $\mathbf{A}$
$D_{n-1}(\lambda)$	greatest common divisor of all the elements of Adj $[\lambda\mathbf{I}_n-\mathbf{A}]$
$L[i\times c]$	multiplication of the i-th row by the number $c\neq 0$
$L[i, j]$	interchange of the i-th and j-th rows
$L[i+j\times b(s)]$	addition of the j-th row multiplied by the polynomial $b(s)$ to the i-th row
$m\times n$	dimension of a matrix with m rows and n columns
$P[i\times c]$	multiplication of the i-th column by the number $c \neq 0$
$P[i, j]$	interchange of the i-th and j-th columns
$P[i+j\times b(s)]$	addition of the j-th column multiplied by the polynomial $b(s)$ to the i-th column
$P[i+j\times b(s)]$	addition of the j-th column multiplied by the polynomial $b(s)$ to the i-th column
tr $\mathbf{A}$	trace of $\mathbf{A}$
rank $\mathbf{A}$	rank of $\mathbf{A}$
Im $\mathbf{A}$	image of $\mathbf{A}$
Ker $\mathbf{A}$	kernel of $\mathbf{A}$
$W(\mathbf{A})$	characteristic polynomial of $\mathbf{A}$
$\varphi(\lambda)$	characteristic polynomial of a matrix
$\Psi(\lambda)$	minimal polynomial of a matrix
λ	eigenvalue

$\mathbf{I}_n$	identity matrix of size n
$\mathbf{0}_n$	zero matrix of size n
M_{ij}	minor of a matrix
$\otimes$	Kronecker product
$\lVert \; \rVert$	norm
$\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$	set of $m\times n$ matrices with entries from the field of complex numbers $\mathbb{C}$, real numbers $\mathbb{R}$
$\mathbb{R}^{m\times n}[s]$	set of $m\times n$ polynomial matrices
$\mathbb{R}^{m\times n}(s)$	set of $m\times n$ rational matrices
$\mathbb{C}[s]$, $\mathbb{R}[s]$	set of polynomials with coefficients from the field $\mathbb{C}$, $\mathbb{R}$
$\mathbb{F}(s)$	field of complex functions of the variable s
$\mathbb{R}_p(s)$	set of rational causal functions with coefficients from the field $\mathbb{R}$
$\mathbb{R}_s(s)$	set of stable rational functions with coefficients form the field $\mathbb{R}$
$\mathbb{R}[s^{-1}]$	set of finite rational functions with coefficients from the field $\mathbb{R}$
$\mathbb{R}_p^{m\times n}(s)$	set of rational causal $m\times n$ matrices with coefficients from the field $\mathbb{R}$
$\mathbb{R}_s^{m\times n}(s)$	set of rational stable $m\times n$ matrices with coefficients from the field $\mathbb{R}$
$\mathbb{R}^{m\times n}[s^{-1}]$	set of rational finite $m\times n$ matrices with the coefficients from the field $\mathbb{R}$
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{W}$	set of rational numbers
$\mathbb{W}(s)$	set of rational functions
$\mathbb{W}[s]$	set of polynomials of the variable s

1

Polynomial Matrices

1.1 Polynomials

Letting $\mathbb{F}$ be a field, e.g., of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the rational numbers $\mathbb{W}$, the rational functions $W(s)$ of a complex variable s, etc.,

$$w(s) = \sum_{i=0}^{n} a_i s^i = a_0 + a_1 s + ... + a_n s^n \tag{1.1.1}$$

is called a polynomial $w(s)$ in the variable s over the field $\mathbb{F}$, where $a_i \in \mathbb{F}$ for $i = 0,1,...,n$ are called the coefficients of this polynomial.

The set of polynomials (1.1.1) over the field $\mathbb{F}$ will be denoted by $\mathbb{F}[s]$.

If $a_n \neq 0$, then the nonnegative integral n is called the degree of a polynomial and is denoted deg $w(s)$, i.e., $n = \deg w(s)$. The polynomial (1.1.1) is called monic, if $a_n = 1$ and zero polynomial, if $a_i = 0$ for $i = 0,1,...,n$. The sum of two polynomials

$$w_1(s) = a_0 + a_1 s + ... + a_n s^n, \tag{1.1.2a}$$

$$w_2(s) = b_0 + b_1 s + ... + b_m s^m, \tag{1.1.2b}$$

is defined in the following way

$$w_1(s) + w_2(s) = \begin{cases} \sum_{i=0}^{m} (a_i + b_i) s^i + \sum_{i=m+1}^{n} a_i s^i, \; n > m \\ \sum_{i=0}^{n} (a_i + b_i) s^i, \; n = m \\ \sum_{i=0}^{n} (a_i + b_i) s^i + \sum_{i=n+1}^{m} b_i s^i, \; m > n \end{cases}. \tag{1.1.3}$$

If $n > m$, then the sum is a polynomial of degree n, if $m > n$ then the sum is a polynomial of degree m. If $n = m$ and $a_n+b_n \neq 0$, then this sum is a polynomial of degree n and a polynomial of degree less than n, if $a_n+b_n = 0$. Thus we have

$$\deg\left[w_1(s)+w_2(s)\right] \leq \max\left[\deg\left[w_1(s)\right], \deg\left[w_2(s)\right]\right]. \tag{1.1.4}$$

In the same vein we define the difference of two polynomials.

A polynomial whose coefficients are the products of the coefficients a_i and the scalar λ, i.e.,

$$\lambda\, w(s) = \sum_{i=0}^{n} \lambda\, a_i s^i \ , \tag{1.1.5}$$

is called the product of the polynomial (1.1.1) and the scalar λ (a scalar can be regarded as a polynomial of zero degree).

A polynomial of the form

$$w_1(s)w_2(s) = \sum_{i=0}^{n+m} c_i s^i \tag{1.1.6a}$$

is called the product of the polynomials (1.1.2), where

$$\begin{gathered} c_i = \sum_{k=0}^{i} a_k b_{i-k}, \quad i = 0, 1, \dots, n+m \\ (a_k = 0 \ \text{ for } k > n, \quad b_k = 0 \ \text{ for } k > m). \end{gathered} \tag{1.1.6b}$$

From (1.1.6a) it follows that

$$\deg\left[w_1(s)w_2(s)\right] = n+m\,, \tag{1.1.7}$$

since $a_n b_m \neq 0$ for $a_n \neq 0$, $b_m \neq 0$.

Let $w_2(s)$ in (1.1.2) be a nonzero polynomial and $n > m$, then there exist exactly two polynomials $q(s)$ and $r(s)$ such that

$$w_1(s) = w_2(s)q(s) + r(s)\,, \tag{1.1.8}$$

where

$$\deg\left[r(s)\right] < \deg\left[w_2(s)\right] = m\,. \tag{1.1.9}$$

The polynomial $q(s)$ is called the integer part when $r(s) \neq 0$ and the quotient when $r(s) = 0$, and $r(s)$ is called the remainder.

If $r(s) = 0$, then $w_1(s) = w_2(s)q(s)$; we say then that polynomial $w_1(s)$ is divisible without remainder by the polynomial $w_2(s)$, or equivalently, that polynomial $w_2(s)$ divides without remainder a polynomial $w_1(s)$, which is denoted by $w_1(s) \mid w_2(s)$. We also say that the polynomial $w_2(s)$ is a divisor of the polynomial $w_1(s)$.

Let us consider the polynomials in (1.1.2). We say that a polynomial $d(s)$ is a common divisor of the polynomials $w_1(s)$ and $w_2(s)$ if there exist polynomials $\overline{w}_1(s)$ and $\overline{w}_2(s)$ such that

$$w_1(s) = d(s)\overline{w}_1(s), \quad w_2(s) = d(s)\overline{w}_2(s) . \tag{1.1.10}$$

Polynomial $d_m(s)$ is called a greatest common divisor (GCD) of the polynomials $w_1(s)$ and $w_2(s)$, if every common divisor of these polynomials is a divisor of the polynomial $d_m(s)$. A GCD $d_m(s)$ of polynomials $w_1(s)$ and $w_2(s)$ is determined uniquely up to multiplication by a constant factor and satisfies the equality

$$d_m(s) = w_1(s)m_1(s) + w_2(s)m_2(s) , \tag{1.1.11}$$

where $m_1(s)$ and $m_2(s)$ are polynomials, which we can determine using Euclid's algorithm or the elementary operations method.

The essence of Euclid's algorithm is as follows. Using division of polynomials we determine the sequences of polynomials $q_1, q_2, \ldots, q_k$ and $r_1, r_2, \ldots, r_k$ satisfying the following properties

$$\left\{\begin{array}{l} w_1 = w_2 q_1 + r_1 \\ w_2 = r_1 q_2 + r_2 \\ r_1 = r_2 q_3 + r_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_{k-2} = r_{k-1} q_k + r_k \\ r_{k-1} = r_k q_{k+1} \end{array}\right\} . \tag{1.1.12}$$

We stop computations when the last nonzero remainder r_k is computed and r_{k-1} is found to be divisible without remainder by r_k. With $r_1, r_2, \ldots, r_{k-1}$ eliminated from (1.1.12) we obtain (1.1.11) for $d_m(s) = r_k$. Thus the last nonzero remainder r_k is a GCD of the polynomials $w_1(s)$ and $w_2(s)$.

Example 1.1.1.
Let

$$w_1 = w_1(s) = s^3 - 3s^2 + 3s - 1, \; w_2 = w_2(s) = s^2 + s + 1 . \tag{1.1.13}$$

Using Euclid's algorithm we compute

$$
\begin{aligned}
&w_1 = w_2 q_1 + r_1,\ \ q_1 = s - 4,\ \ r_1 = 6s + 3,\\
&w_2 = r_1 q_2 + r_2,\ \ q_2 = \frac{1}{6}s + \frac{1}{12},\ \ r_2 = \frac{3}{4}.
\end{aligned}
\tag{1.1.14}
$$

Here we stop because r_1 is divisible without remainder by r_2.

Thus r_2 is a GCD of the polynomials in (1.1.13). Elimination of r_1 from (1.1.14) yields

$$
w_1(-q_2) + w_2(1 + q_1 q_2) = r_2 ,
$$

that is,

$$
\left(-s^3 + 3s^2 - 3s + 1\right)\left(\frac{1}{6}s + \frac{1}{12}\right) + \left(s^2 + s + 1\right)\left(\frac{1}{6}s^2 - \frac{7}{12}s + \frac{2}{3}\right) = \frac{3}{4}.
$$

The polynomials in (1.1.2) are called relatively prime (or coprime) if and only if their monic GCD is equal to 1. From (1.1.11) for $d_m(s) = 1$ it follows that polynomials $w_1(s)$ and $w_2(s)$ are coprime if and only if there exist polynomials $m_1(s)$ and $m_2(s)$ such that

$$
w_1(s)m_1(s) + w_2(s)m_2(s) = 1 .
\tag{1.1.15}
$$

Dividing both sides of (1.1.11) by $d_m(s)$, we obtain

$$
1 = \widehat{w}_1(s)m_1(s) + \widehat{w}_2(s)m_2(s) ,
\tag{1.1.16}
$$

where

$$
\widehat{w}_k(s) = \frac{w_k(s)}{d_m(s)} \quad \text{for} \quad k = 1,2,\ldots .
$$

Thus if $d_m(s)$ is a GCD of the polynomials $w_1(s)$ and $w_2(s)$, then polynomials $\widehat{w}_1(s)$ and $\widehat{w}_2(s)$ are coprime.

Let $s_1, s_2, \ldots, s_p$ be different roots of multiplicities $m_1, m_2, \ldots, m_p$ ($m_1 + m_2 + \ldots + m_p = n$), respectively, of the equation $w(s) = 0$. The numbers $s_1, s_2, \ldots, s_p$ are called the zeros of polynomial (1.1.1). This polynomial can be uniquely written in the form

$$
w(s) = a_n(s - s_1)^{m_1}(s - s_2)^{m_2} \ldots (s - s_p)^{m_p} .
\tag{1.1.17}
$$

1.2 Basic Notions and Basic Operations on Polynomial Matrices

A matrix whose elements are polynomials over a field $\mathbb{F}$ is called a polynomial matrix over the field $\mathbb{F}$ (briefly polynomial matrix)

$$\mathbf{A}(s) = \left[a_{ij}(s)\right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \begin{bmatrix} a_{11}(s) & \dots & a_{1n}(s) \\ \vdots & \ddots & \vdots \\ a_{m1}(s) & \dots & a_{mn}(s) \end{bmatrix}, \quad a_{ij}(s) \in \mathbb{F}(s). \tag{1.2.1}$$

An ordered pair of the number of rows m and columns n, respectively, is called the dimension of matrix (1.2.1) and is denoted by $m \times n$. A set of polynomial matrices of dimension $m \times n$ over a field $\mathbb{F}$ will be denoted by $\mathbb{F}^{m\times n}[s]$.

The following matrix is an example of a 2×2 polynomial matrix over the field of real numbers

$$\mathbf{A}_0(s) = \begin{bmatrix} s^2 + 2s + 1 & s + 2 \\ 2s^2 + s + 3 & 3s^2 + s - 3 \end{bmatrix} \in \mathbb{R}^{2\times 2}[s]. \tag{1.2.2}$$

Every polynomial matrix can be written in the form of a matrix polynomial. For example, the matrix (1.2.2) can be written in the form of the matrix polynomial

$$\mathbf{A}_0(s) = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} s^2 + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix} = \mathbf{A}_2 s^2 + \mathbf{A}_1 s + \mathbf{A}_0. \tag{1.2.3}$$

Let a matrix of the form (1.2.1) be expressed as the matrix polynomial

$$\mathbf{A}(s) = \mathbf{A}_q s^q + \dots + \mathbf{A}_1 s + \mathbf{A}_0, \ \mathbf{A}_k \in \mathbb{R}^{m\times n}, \quad k = 0, 1, \dots, q. \tag{1.2.4}$$

If $\mathbf{A}_q$ is not a zero matrix, then number q is called its degree and is denoted by $q = \deg \mathbf{A}(s)$. For example, the matrix (1.2.2) (and also (1.2.3)) has the degree two $q = 2$.

If $n = m$ and $\det \mathbf{A}_q \neq 0$, then matrix (1.2.4) is called regular.

The sum of two polynomial matrices

$$\begin{aligned} \mathbf{A}(s) &= \left[a_{ij}(s)\right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \sum_{k=0}^{q} \mathbf{A}_k s^k \quad \text{and} \\ \mathbf{B}(s) &= \left[b_{ij}(s)\right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \sum_{k=0}^{t} \mathbf{B}_k s^k \end{aligned} \tag{1.2.5}$$

of the same dimension $m \times n$ is defined in the following way

$$\mathbf{A}(s)+\mathbf{B}(s)=$$

$$=\left[a_{ij}(s)+b_{ij}(s)\right]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}=\begin{cases}\sum_{k=0}^{t}(\mathbf{A}_k+\mathbf{B}_k)s^k+\sum_{k=t+1}^{q}\mathbf{A}_k s^k & q>t\\ \sum_{k=0}^{q}(\mathbf{A}_k+\mathbf{B}_k)s^k & q=t\\ \sum_{k=0}^{q}(\mathbf{A}_k+\mathbf{B}_k)s^k+\sum_{k=q+1}^{t}\mathbf{B}_k s^k & q<t\end{cases}. \tag{1.2.6}$$

If $q = t$ and $\mathbf{A}_q + \mathbf{B}_q \neq 0$, then the sum in (1.2.6) is a polynomial matrix of degree q, and if $\mathbf{A}_q + \mathbf{B}_q = 0$, then this sum is a polynomial matrix of a degree not greater than q. Thus we have

$$\deg\left[\mathbf{A}(s)+\mathbf{B}(s)\right]\le \max\left[\deg\left[\mathbf{A}(s)\right],\deg\left[\mathbf{B}(s)\right]\right]. \tag{1.2.7}$$

In the same vein, we define the difference of two polynomial matrices.

A polynomial matrix where every entry is the product of an entry of the matrix (1.2.1) and the scalar λ is called the product of the polynomial matrix (1.2.1) and the scalar λ

$$\lambda\,\mathbf{A}(s)=\left[\lambda\, a_{ij}(s)\right]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} .$$

From this definition for $\lambda \neq 0$, we have $\deg\,[\lambda\mathbf{A}(s)] = \deg\,[\mathbf{A}(s)]$.

Multiplication of two polynomial matrices can be carried out if and only if the number of columns of the first matrix (1.2.1) is equal to the number of rows of the second matrix

$$\mathbf{B}(s)=\left[b_{ij}(s)\right]_{\substack{i=1,\dots,n\\ j=1,\dots,p}}=\sum_{k=0}^{t}\mathbf{B}_k s^k . \tag{1.2.8}$$

A polynomial matrix of the form

$$\mathbf{C}(s)=\left[c_{ij}(s)\right]_{\substack{i=1,\dots,m\\ j=1,\dots,p}}=\mathbf{A}(s)\mathbf{B}(s)=\sum_{k=0}^{q+t}\mathbf{C}_k s^k \tag{1.2.9}$$

is called the product of these polynomial matrices, where

$$\begin{gathered}\mathbf{C}_k=\sum_{l=0}^{k}\mathbf{A}_l\mathbf{B}_{k-l}\quad k=0,1,\dots,q+t\\ (\mathbf{A}_l=0,\ l>q,\ \mathbf{B}_l=0,\ l>t)\ .\end{gathered} \tag{1.2.10}$$