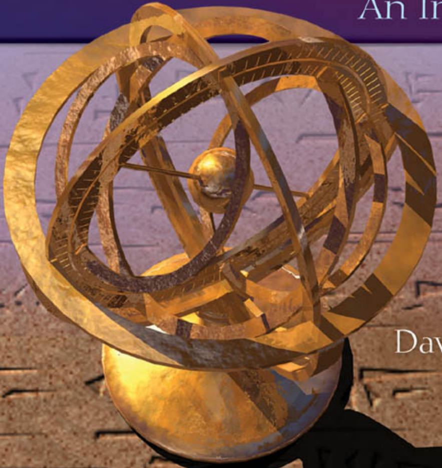
Seventh Edition

THE HISTORY *of* MATHEMATICS

An Introduction



David M. Burton

The History of Mathematics

AN INTRODUCTION

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THE HISTORY OF MATHEMATICS: AN INTRODUCTION, SEVENTH EDITION

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All these were honored in their generations, and were the glory of their times.

There be of them, that have left a name behind them, that their praises might be reported.

And some there be, which have no memorial; who are perished, as though they had never been; and are become as though they had never been born; and their children after them.

ECCLESIASTICUS 44:7–9

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Preface

Since many excellent treatises on the history of mathematics are available, there may seem to be little reason for writing another. But most current works are severely technical, written by mathematicians for other mathematicians or for historians of science. Despite the admirable scholarship and often clear presentation of these works, they are not especially well adapted to the undergraduate classroom. (Perhaps the most notable exception is Howard Eves's popular account, *An Introduction to the History of Mathematics*.) There is a need today for an undergraduate textbook, which is also accessible to the general reader interested in the history of mathematics. In the following pages, I have tried to give a reasonably full account of how mathematics has developed over the past 5000 years. Because mathematics is one of the oldest intellectual instruments, it has a long story, interwoven with striking personalities and outstanding achievements. This narrative is chronological, beginning with the origin of mathematics in the great civilizations of antiquity and progressing through the later decades of the twentieth century. The presentation necessarily becomes less complete for modern times, when the pace of discovery has been rapid and the subject matter more technical.

Considerable prominence has been assigned to the lives of the people responsible for progress in the mathematical enterprise. In emphasizing the biographical element, I can say only that there is no sphere in which individuals count for more than the intellectual life, and that most of the mathematicians cited here really did tower over their contemporaries. So that they will stand out as living figures and representatives of their day, it is necessary to pause from time to time to consider the social and cultural framework in which they lived. I have especially tried to define why mathematical activity waxed and waned in different periods and in different countries.

Writers on the history of mathematics tend to be trapped between the desire to interject some genuine mathematics into a work and the desire to make the reading as painless and pleasant as possible. Believing that any mathematics textbook should concern itself primarily with teaching mathematical content, I have favored stressing the mathematics. Thus, assorted problems of varying degrees of difficulty have been interspersed throughout. Usually these problems typify a particular historical period, requiring the procedures of that time. They are an integral part of the text and, in working them, you will learn some interesting mathematics as well as history. The level of maturity needed for this work is approximately the mathematical background of a college junior or senior. Readers with more extensive training in the subject must forgive certain explanations that seem unnecessary.

The title indicates that this book is in no way an encyclopedic enterprise: it does not pretend to present all the important mathematical ideas that arose during the vast sweep of time it covers. The inevitable limitations of space necessitate illuminating some outstanding landmarks instead of casting light of equal brilliance over the whole landscape. A certain amount of judgment and self-denial has been exercised, both in choosing mathematicians and in treating their contributions. The material that appears here does reflect some personal tastes and prejudices. It stands to reason that not everyone will be satisfied with the choices. Some readers will raise an eyebrow at the omission of some household names of mathematics that have been either passed over in complete silence or shown no great hospitality; others will regard the scant treatment of their favorite topic as an unpardonable omission. Nevertheless, the path that I have pieced together should provide an adequate explanation of how mathematics came to occupy its position as a primary cultural force in Western civilization. The book is published in the modest hope that it may stimulate the reader to pursue more elaborate works on the subject.

Anyone who ranges over such a well-cultivated eld as the history of mathematics becomes much indebted to the scholarship of others. The chapter bibliographies represent a partial listing of works that in one way or another have helped my command of the facts. To the writers and many others of whom no record was kept, I am enormously grateful.

New to This Edition

Readers familiar with previous editions of *The History of Mathematics* will nd that this seventh edition maintains the same overall organization and content. Nevertheless, the preparation of a seventh edition has provided the occasion for a variety of small improvements as well as several more signi cant ones.

The most notable difference is an enhanced treatment of American mathematics. Section 12.1, for instance, includes the efforts of such early nineteenth-century gures as Robert Adrain and Benjamin Banneker. Because the mathematically gifted of the period often became observational astronomers, the contributions of Simon Newcomb, George William Hill, Albert Michelson, and Maria Mitchell are also recounted. Later sections consider the work of more recent mathematicians, such as Oswald Veblen, R. L. Moore, Richard Courant, and Walter Feit.

Another noteworthy difference is the attention now paid to several mathematicians passed over in previous editions. Among them are Lazar Carnot, Herman Günther Grassmann, Andrei Kolmogorov, William Burnside, and Paul Erdös.

Beyond these modi cations, there are some minor changes: biographies are brought up to date and certain numerical information kept current. In addition, an attempt has been made to correct errors, both typographical and historical, which crept into the earlier editions.

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Acknowledgments

Many friends, colleagues, and readers—too numerous to mention individually—have been kind enough to forward corrections or to offer suggestions for the book's enrichment. My thanks to all for their collective contributions. Although not every recommendation was incorporated, all were gratefully received and seriously considered when deciding upon alterations.

In particular, the advice of the following reviewers was especially helpful in the creation of the seventh edition:

Victor Akatsa, *Chicago State University*
Carl FitzGerald, *The University of California, San Diego*
Gary Shannon, *California State University, Sacramento*
Tomas Smotzer, *Youngstown State University*
John Stroyls, *Georgia Southwestern State University*

A special debt of thanks is owed my wife, Martha Beck Burton, for providing assistance throughout the preparation of this edition. Her thoughtful comments signi cantly improved the exposition. Finally, I would like to express my appreciation to the staff members of McGraw-Hill for their unfailing cooperation during the course of production.

Any errors that have survived all this generous assistance must be laid at my door.

D. M. B.

CHAPTER 1

Early Number Systems and Symbols

To think the thinkable—that is the mathematician's aim.

C.J. KEYSER

1.1 Primitive Counting

A Sense of Number

The root of the term *mathematics* is in the Greek word *mathemata*, which was used quite generally in early writings to indicate any subject of instruction or study. As learning advanced, it was found convenient to restrict the scope of this term to particular fields of knowledge. The Pythagoreans are said to have used it to describe arithmetic and geometry; previously, each of these subjects had been called by its separate name, with no designation common to both. The Pythagoreans' use of the name would perhaps be a basis for the notion that mathematics began in Classical Greece during the years from 600 to 300 B.C. But its history can be followed much further back. Three or four thousand years ago, in ancient Egypt and Babylonia, there already existed a significant body of knowledge that we should describe as mathematics. If we take the broad view that mathematics involves the study of issues of a quantitative or spatial nature—number, size, order, and form—it is an activity that has been present from the earliest days of human experience. In every time and culture, there have been people with a compelling desire to comprehend and master the form of the natural world around them. To use Alexander Pope's words, "This mighty maze is not without a plan."

It is commonly accepted that mathematics originated with the practical problems of counting and recording numbers. The birth of the idea of number is so hidden behind the veil of countless ages that it is tantalizing to speculate on the remaining evidences of early humans' sense of number. Our remote ancestors of some 20,000 years ago—who were quite as clever as we are—must have felt the need to enumerate their livestock, tally objects for barter, or mark the passage of days. But the evolution of counting, with its spoken number words and written number symbols, was gradual and does not allow any determination of precise dates for its stages.

Anthropologists tell us that there has hardly been a culture, however primitive, that has not had some awareness of number, though it might have been as rudimentary as the distinction between one and two. Certain Australian aboriginal tribes, for instance, counted to two only, with any number larger than two called simply "much" or "many." South American Indians along the tributaries of the Amazon were equally destitute of number words. Although they ventured further than the aborigines in being able to count to six, they had no independent number names for groups of three, four, five, or six. In their counting vocabulary, three was called "two-one," four was "two-two," and so on. A similar system has been reported for the Bushmen of South Africa, who counted to ten ($10 = 2 + 2 + 2 + 2 + 2$) with just two words; beyond ten, the descriptive phrases became too long. It is notable that such tribal groups would not willingly trade, say, two cows for four pigs, yet had no hesitation in exchanging one cow for two pigs and a second cow for another two pigs.

The earliest and most immediate technique for visibly expressing the idea of number is tallying. The idea in tallying is to match the collection to be counted with some easily employed set of objects—in the case of our early forebears, these were fingers, shells, or stones. Sheep, for instance, could be counted by driving them one by one through a narrow passage while dropping a pebble for each. As the flock was gathered in for the night, the pebbles were moved from one pile to another until all the sheep had been accounted for. On the occasion of a victory, a treaty, or the founding of a village, frequently a cairn, or pillar of stones, was erected with one stone for each person present.

The term *tally* comes from the French verb *tailler*, "to cut," like the English word *tailor*; the root is seen in the Latin *taliare*, meaning "to cut." It is also interesting to note that the English word *write* can be traced to the Anglo-Saxon *writan*, "to scratch," or "to notch."

Neither the spoken numbers nor finger tallying have any permanence, although finger counting shares the visual quality of written numerals. To preserve the record of any count, it was necessary to have other representations. We should recognize as human intellectual progress the idea of making a correspondence between the events or objects recorded and a series of marks on some suitably permanent material, with one mark representing each individual item. The change from counting by assembling collections of physical objects to counting by making collections of marks on one object is a long step, not only toward abstract number concept, but also toward written communication.

Counts were maintained by making scratches on stones, by cutting notches in wooden sticks or pieces of bone, or by tying knots in strings of different colors or lengths. When the numbers of tally marks became too unwieldy to visualize, primitive people arranged them in easily recognizable groups such as groups of 5, for the fingers of a hand. It is likely that grouping by pairs came first, soon abandoned in favor of groups of 5, 10, or 20. The organization of counting by groups was a noteworthy improvement on counting by ones. The practice of counting by fives, say, shows a tentative sort of progress toward reaching an abstract concept of "five" as contrasted with the descriptive ideas "five fingers" or "five days." To be sure, it was a timid step in the long journey toward detaching the number sequence from the objects being counted.

Notches as Tally Marks

Bone artifacts bearing incised markings seem to indicate that the people of the Old Stone Age had devised a system of tallying by groups as early as 30,000 B.C. The most impressive example is a shinbone from a young wolf, found in Czechoslovakia in 1937; about 7 inches long, the bone is engraved with 55 deeply cut notches, more or less equal in length, arranged in groups of five. (Similar recording notations are still used, with the strokes bundled in fives, like 𝍸. Voting results in small towns are still counted in the manner devised by our remote ancestors.) For many years such notched bones were interpreted as hunting tallies and the incisions were thought to represent kills. A more recent theory, however, is that the first recordings of ancient people were concerned with reckoning time. The markings on bones discovered in French cave sites in the late 1880s are grouped in sequences of recurring numbers that agree with the numbers of days included in successive phases of the moon. One might argue that these incised bones represent lunar calendars.

Another arresting example of an incised bone was unearthed at Ishango along the shores of Lake Edward, one of the headwater sources of the Nile. The best archeological and geological evidence dates the site to 17,500 B.C., or some 12,000 years before the first settled agrarian communities appeared in the Nile valley. This fossil fragment was probably the handle of a tool used for engraving, or tattooing, or even writing in some way. It contains groups of notches arranged in three definite columns; the odd, unbalanced composition does not seem to be decorative. In one of the columns, the groups are composed of 11, 21, 19, and 9 notches. The underlying pattern may be $10 + 1$, $20 + 1$, $20 - 1$, and $10 - 1$. The notches in another column occur in eight groups, in the following order: 3, 6, 4, 8, 10, 5, 5, 7. This arrangement seems to suggest an appreciation of the concept of duplication, or multiplying by 2. The last column has four groups consisting of 11, 13, 17, and 19 individual notches. The pattern here may be fortuitous and does not necessarily indicate—as some authorities are wont to infer—a familiarity with prime numbers. Because $11 + 13 + 17 + 19 = 60$ and $11 + 21 + 19 + 9 = 60$, it might be argued that markings on the prehistoric Ishango bone are related to a lunar count, with the first and third columns indicating two lunar months.

The use of tally marks to record counts was prominent among the prehistoric peoples of the Near East. Archaeological excavations have unearthed a large number of small clay objects that had been hardened by fire to make them more durable. These handmade artifacts occur in a variety of geometric shapes, the most common being circular disks, triangles, and cones. The oldest, dating to about 8000 B.C., are incised with sets of parallel lines on a plain surface; occasionally, there will be a cluster of circular impressions as if punched into the clay by the blunt end of a bone or stylus. Because they go back to the time when people first adopted a settled agricultural life, it is believed that the objects are primitive reckoning devices; hence, they have become known as "counters" or "tokens." It is quite likely also that the shapes represent different commodities. For instance, a token of a particular type might be used to indicate the number of animals in a herd, while one of another kind could count measures of grain. Over several millennia, tokens became increasingly complex, with diverse markings and new shapes. Eventually, there came to be 16 main forms of tokens. Many were perforated with small holes, allowing them to be strung together for safekeeping. The token system of recording information went out of favor around 3000 B.C., with the rapid adoption of writing on clay tablets.

A method of tallying that has been used in many different times and places involves the notched stick. Although this device provided one of the earliest forms of keeping records, its use was by no means limited to "primitive peoples," or for that matter, to the remote past. The acceptance of tally sticks as promissory notes or bills of exchange reached its highest level of development in the British Exchequer tallies, which formed an essential part of the government records from the twelfth century onward. In this instance, the tallies were flat pieces of hazelwood about 6–9 inches long and up to an inch thick. Notches of varying sizes and types were cut in the tallies, each notch representing a fixed amount of money. The width of the cut decided its value. For example, the notch of £1000 was as large as the width of a hand; for £100, as large as the thickness of a thumb; and for £20, the width of the little finger. When a loan was made the appropriate notches were cut and the stick split into two pieces so that the notches appeared in each section. The debtor kept one piece and the Exchequer kept the other, so the transaction could easily be verified by fitting the two halves together and noticing whether the notches coincided (whence the expression "our accounts tallied"). Presumably, when the two halves had been matched, the Exchequer destroyed its section—either by burning it or by making it smooth again by cutting off the notches—but retained the debtor's section for future record. Obstinate adherence to custom kept this wooden accounting system in official use long after the rise of banking institutions and modern numeration had made its practice quaintly obsolete. It took an act of Parliament, which went into effect in 1826, to abolish the practice. In 1834, when the long-accumulated tallies were burned in the furnaces that heated the House of Lords, the fire got out of hand, starting a more general conflagration that destroyed the old Houses of Parliament.

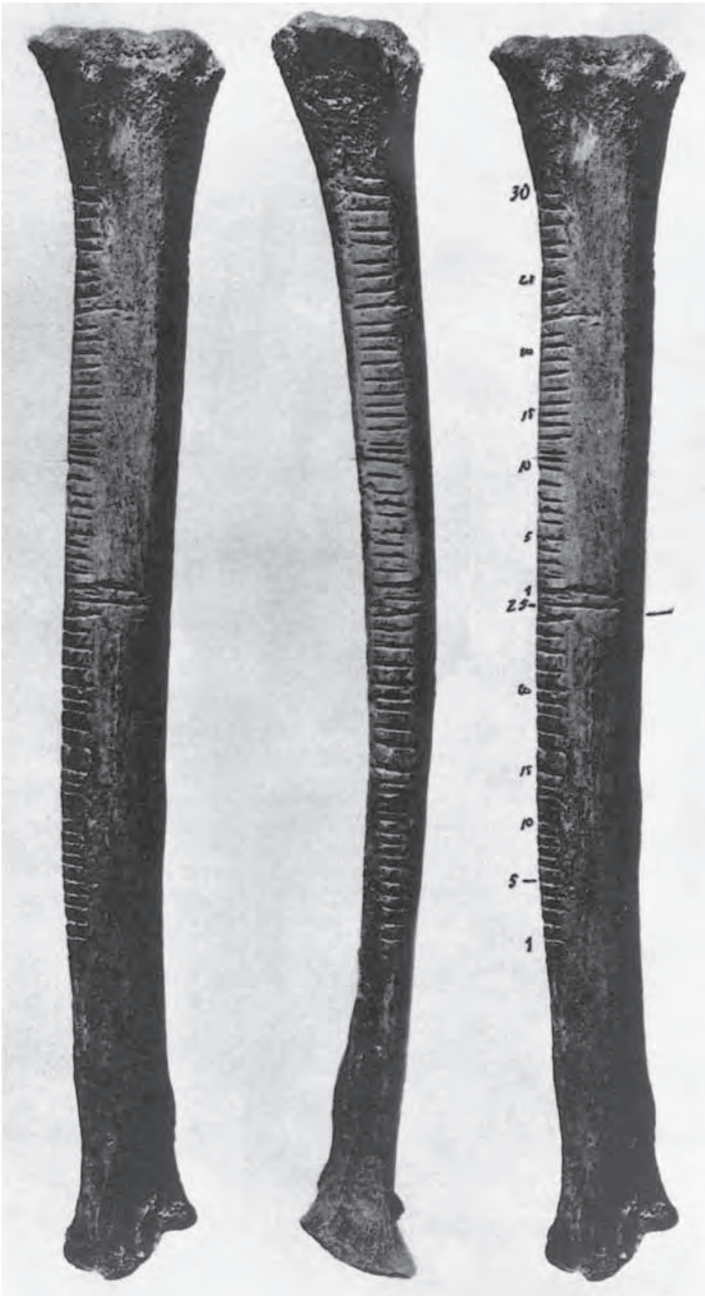
The English language has taken note of the peculiar quality of the double tally stick. Formerly, if someone lent money to the Bank of England, the amount was cut on a tally stick, which was then split. The piece retained by the bank was known as the foil, whereas the other half, known as the stock, was given the lender as a receipt for the sum of money paid in. Thus, he became a "stockholder" and owned "bank stock" having the same worth as paper money issued by the government. When the holder would return, the stock was carefully checked and compared against the foil in the bank's possession; if they agreed, the owner's piece would be redeemed in currency. Hence, a written certificate that was presented for remittance and checked against its security later came to be called a "check."

Using wooden tallies for records of obligations was common in most European countries and continued there until fairly recently. Early in this century, for instance, in some remote valleys of Switzerland, "milk sticks" provided evidence of transactions among farmers who owned cows in a common herd. Each day the chief herdsman would carve a six- or seven-sided rod of ashwood, coloring it with red chalk so that incised lines would stand out vividly. Below the personal symbol of each farmer, the herdsman marked off the amounts of milk, butter, and cheese yielded by a farmer's cows. Every Sunday after church, all parties would meet and settle the accounts. Tally sticks—in particular, double tallies—were recognized as legally valid documents until well into the 1800s. France's first modern code of law, the Code Civil, promulgated by Napoleon in 1804, contained the provision:

> The tally sticks which match their stocks have the force of contracts between persons who are accustomed to declare in this manner the deliveries they have made or received.

The variety in practical methods of tallying is so great that giving any detailed account would be impossible here. But the procedure of counting both days and objects by means of knots tied in cords has such a long tradition that it is worth mentioning. The device was frequently used in ancient Greece, and we find reference to it in the work of Herodotus (fifth century B.C.). Commenting in his *History*, he informs us that the Persian king Darius handed the Ionians a knotted cord to serve as a calendar:

> The King took a leather thong and tying sixty knots in it called together the Ionian tyrants and spoke thus to them: "Untie every day one of the knots; if I do not return before the



Three views of a Paleolithic wolfbone used for tallying. *(The Illustrated London News Picture Library.)*

> last day to which the knots will hold out, then leave your station and return to your several homes."

The Peruvian Quipus: Knots as Numbers

In the New World, the number string is best illustrated by the knotted cords, called *quipus*, of the Incas of Peru. They were originally a South American Indian tribe, or a collection of kindred tribes, living in the central Andean mountainous highlands. Through gradual expansion and warfare, they came to rule a vast empire consisting of the coastal and mountain regions of present-day Ecuador, Peru, Bolivia, and the northern parts of Chile and Argentina. The Incas became renowned for their engineering skills, constructing stone temples and public buildings of a great size. A striking accomplishment was their creation of a vast network (as much as 14,000 miles) of roads and bridges linking the far-flung parts of the empire. The isolation of the Incas from the horrors of the Spanish Conquest ended early in 1532 when 180 conquistadors landed in northern Peru. By the end of the year, the invaders had seized the capital city of Cuzco and imprisoned the emperor. The Spaniards imposed a way of life on the people that within about 40 years would destroy the Inca culture.

When the Spanish conquerors arrived in the sixteenth century, they observed that each city in Peru had an "official of the knots," who maintained complex accounts by means of knots and loops in strands of various colors. Performing duties not unlike those of the city treasurer of today, the quipu keepers recorded all official transactions concerning the land and subjects of the city and submitted the strings to the central government in Cuzco. The quipus were important in the Inca Empire, because apart from these knots no system of writing was ever developed there. The quipu was made of a thick main cord or crossbar to which were attached finer cords of different lengths and colors; ordinarily the cords hung down like the strands of a mop. Each of the pendent strings represented a certain item to be tallied; one might be used to show the number of sheep, for instance, another for goats, and a third for lambs. The knots themselves indicated numbers, the values of which varied according to the type of knot used and its specific position on the strand. A decimal system was used, with the knot representing units placed nearest the bottom, the tens appearing immediately above, then the hundreds, and so on; absence of a knot denoted zero. Bunches of cords were tied off by a single main thread, a summation cord, whose knots gave the total count for each bunch. The range of possibilities for numerical representation in the quipus allowed the Incas to keep incredibly detailed administrative records, despite their ignorance of the written word. More recent (1872) evidence of knots as a counting device occurs in India; some of the Santal headsmen, being illiterate, made knots in strings of four different colors to maintain an up-to-date census.

To appreciate the quipu fully, we should notice the numerical values represented by the tied knots. Just three types of knots were used: a figure-eight knot standing for 1, a long knot denoting one of the values 2 through 9, depending on the number of twists in the knot, and a single knot also indicating 1. The figure-eight knot and long knot appear only in the lowest (units) position on a cord, while clusters of single knots can appear in the other spaced positions. Because pendant cords have the same length, an empty position (a value of zero) would be apparent on comparison with adjacent cords.

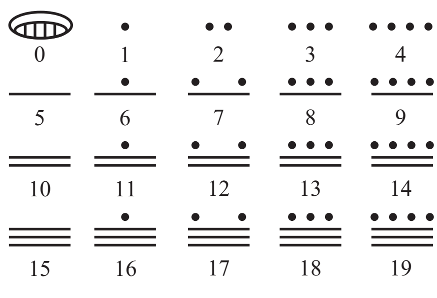
Also, the reappearance of either a figure-eight or long knot would point out that another number is being recorded on the same cord.

Recalling that ascending positions carry place value for successive powers of ten, let us suppose that a particular cord contains the following, in order: a long knot with four twists, two single knots, an empty space, seven clustered single knots, and one single knot. For the Inca, this array would represent the number

$$17024 = 4 + (2 \cdot 10) + (0 \cdot 10^2) + (7 \cdot 10^3) + (1 \cdot 10^4).$$

Another New World culture that used a place value numeration system was that of the ancient Maya. The people occupied a broad expanse of territory embracing southern Mexico and parts of what is today Guatemala, El Salvador, and Honduras. The Mayan civilization existed for over 2000 years, with the time of its greatest flowering being the period 300–900 A.D. A distinctive accomplishment was its development of an elaborate form of hieroglyphic writing using about 1000 glyphs. The glyphs are sometimes sound based and sometimes meaning based: the vast majority of those that have survived have yet to be deciphered. After 900 A.D., the Mayan civilization underwent a sudden decline—The Great Collapse—as its populous cities were abandoned. The cause of this catastrophic exodus is a continuing mystery, despite speculative explanations of natural disasters, epidemic diseases, and conquering warfare. What remained of the traditional culture did not succumb easily or quickly to the Spanish Conquest, which began shortly after 1500. It was a struggle of relentless brutality, stretching over nearly a century, before the last unconquered Mayan kingdom fell in 1597.

The Mayan calendar year was composed of 365 days divided into 18 months of 20 days each, with a residual period of 5 days. This led to the adoption of a counting system based on 20 (a vigesimal system). Numbers were expressed symbolically in two forms. The priestly class employed elaborate glyphs of grotesque faces of deities to indicate the numbers 1 through 19. These were used for dates carved in stone, commemorating notable events. The common people recorded the same numbers with combinations of bars and dots, where a short horizontal bar represented 5 and a dot 1. A particular feature was a stylized shell that served as a symbol for zero; this is the earliest known use of a mark for that number.



The symbols representing numbers larger than 19 were arranged in a vertical column with those in each position, moving upward, multiplied by successive powers of 20; that is, by 1, 20, 400, 8000, 160,000, and so on. A shell placed in a position would indicate the absence of bars and dots there. In particular, the number 20 was expressed by a shell at the bottom of the column and a single dot in the second position. For an example