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Editors

Multimedia Tools for Communicating Mathematics

With 114 Figures, 46 in Color

With CD-ROM

Springer

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Preface

This book on multimedia tools for communicating mathematics arose from presentations at an international workshop organized at the Centro de Matemática e Aplicações Fundamentais at the University of Lisbon, in November 2000, with the collaboration of the Sonderforschungsbereich 288 at the University of Technology in Berlin, and of the Centre for Experimental and Constructive Mathematics at Simon Fraser University in Burnaby, Canada. The MTCM2000 meeting aimed at the scientific methods and algorithms at work inside multimedia tools, and it provided an overview of the range of present multimedia projects, of their limitations and the underlying mathematical problems. The workshop gathered fifty seven participants, twenty nine presentations and a round table. It took place under the auspices of the Sociedade Portuguesa de Matematica and the European Mathematical Society, and was sponsored by a special grant from the Fundação para a Ciência e a Tecnologia of Portugal.

This book presents some of the tools and algorithms currently being used to create new ways of making enhanced interactive presentations and experiments. It is, we hope, an invaluable and up-to-date reference book on multimedia tools presently available for mathematics and related subjects.

The current sources for mathematical knowledge are still largely classical journals and books, even if they are now often available from an electronic archive such as the Los Alamos Server. Nevertheless, a number of new online sources have appeared and hint at what is on the horizon: for example, Neil Sloane's *server of integer sequences*, *Finch's Constants* at MathSoft, or the newly established *EG-Models server* in Berlin with its peer-refereed geometry models. Many people are making or have made large collections of varied mathematical resources of potential interest for a broad mathematical community. The internet has appeared in full battle dress and allows individuals to make such material widely accessible on common platforms.

Currently, many tools and projects focus on the enhancement of digital publications aiming to provide interactive research, experiments and teaching tools online. As of yet they provide limited functionality. We believe that the diversity of multimedia tools for the doing of mathematics will grow substantially in the near future and will profoundly effect the way mathematicians do mathematics.

This was also a general outcome of the lively and participated round table held at MTCM2000. Co-ordinated by J.F. Rodrigues, the discussion concentrated on four main topics, each one addressed by an invited participant: business models for multimedia tools (R. Fitzgerald and J. Borwein), new online services to provide mathematical knowledge (T. Banchoff), new mathematical algorithms and data structures for online mathematics (K. Polthier), and multimedia tools of the future (J. Richter-Gebert).

Besides new tools, new mathematical algorithms and data structures are needed for doing mathematics online. Although in the near future bandwidth will increase dramatically and will open unforeseen possibilities for creative people, the net will still limit the size of experiments, much as today's 3d experiments in numerical mathematics are limited by available computer memory. For instance, as a consequence reduction of the amount of unnecessary data transferred during experiments will continue to be a central research issue.

We hope that the methods and tools discussed in this book and the accompanying CD will provide fruitful and stimulating ground for the further development of multimedia tools for mathematical education, communication and research.

November 2001

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1 Computer Animated Mathematics Videotapes

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1.1 Introduction

Visualization – the representation of ideas, principles or problems by images – has always played an important role in both teaching and learning mathematics. Visual images make a much greater impact than printed or spoken words. People tend to forget words that they read or hear, but images are retained for a long time because they have emotional as well as intellectual appeal. This is especially true when the images are in motion and are accompanied by music and sound effects. The impact of well-crafted televised images on the human mind has been exploited by entertainers, advertisers, and politicians since the advent of television.

In recent years the full power of video technology has also been used to provide a valuable pedagogical tool for attracting young students to mathematics by showing it to be understandable, exciting, and eminently worthwhile. The secret weapon is computer animation designed and executed by James F. Blinn who for many years produced planetary flyby simulations for NASA at the Jet Propulsion Laboratory. In the 1980s Blinn created more than seven hours of computer animation for the 52-episode physics telecourse *The Mechanical Universe and Beyond* produced jointly by the California Institute of Technology (Caltech) and the Southern California Consortium. In the 1990s he extended this pioneering work as part of *Project MATHEMATICS!*, another Caltech effort described below.

The combination of computer animation and video provides a powerful instructional aid that: (a) grabs the viewer's attention and maintains the viewer's involvement; (b) capitalizes on the viewer's geometric intuition; (c) portrays a large quantity and diversity of information in a brief period of time; (d) takes advantage of the viewer's sophistication in reading visual clues; and (e) conveys mathematics in a rich cultural context.

1.2 *Project MATHEMATICS!*

The author of this article, together with James Blinn, launched *Project MATHEMATICS!* in 1987. By the year 2000 the project had produced ten broadcast quality videotapes used as support material in high school and

J. Borwein et al. (eds.), *Multimedia Tools for Communicating Mathematics*

community college classrooms. Each tape, less than 30 minutes in length, is accompanied by a workbook/program guide. Major financial support for the first nine tapes was provided by the National Science Foundation, with additional support from The Educational Foundation of America, SIGGRAPH, Hewlett-Packard, and the Intel Foundation. The tenth videotape, *Early History of Mathematics*, was funded by Caltech Distinguished Alumnus Dr. Irving S. Reed, one of the developers of the Reed-Solomon code.

More than 140,000 copies of *Project MATHEMATICS!* videotapes were in circulation by year 2000, and at least 10 million students had seen one or more of the programs. The actual numbers are undoubtedly much higher because the tapes can be copied freely in the USA for educational purposes. The modules captured a dozen prestigious awards for excellence in educational video production. The tapes are also being broadcast by an ever-growing informal ITV alliance of more than 30 PBS stations, as well as over the NASA Select Television Network. The materials are distributed in Australia, Canada, Denmark, and Great Britain. Eight modules were translated into Hebrew for broadcast on Israeli Educational Television, and also in Portuguese by the University of Lisbon Center of Mathematics and Fundamental Applications.

The titles of the programs, listed in the order produced to date, are *The Theorem of Pythagoras, The Story of Pi, Similarity, Polynomials, Teachers Workshop, Sines and Cosines, Parts I, II,* and *III, The Tunnel of Samos,* and *Early History of Mathematics.*

An earlier article [1] described some of the visualization techniques employed in the first four videotapes of the series. This article does the same for the last five videotapes.

1.3 The Videotape *Sines and Cosines, Part I*

1.3.1 Brief Outline of the Program

After a brief review of prerequisites dealing with properties of similar figures and the number π, the program opens with examples of circular motion in real life, and then introduces the sine in connection with a point moving counterclockwise on a circle of unit radius. The distance the point moves along the circumference is the radian measure of the corresponding central angle and is recorded on a horizontal t axis. The height y of the moving point above or below the horizontal diameter is called the sine of t, written $y = \sin t$. When y is plotted against t the resulting graph is called a sine curve or a sine wave.

By reflecting the sine curve about various lines, some simple properties of the sine are revealed, for example, $\sin(\pi - t) = \sin t$, $\sin(\pi + t) = -\sin t$, and $\sin(-t) = -\sin t$. Reflecting the sine curve about the line $t = \pi/4$ generates a new curve, called a cosine curve, given by $\cos t = \sin(\pi/2 - t)$. The cosine curve is also obtained by shifting the sine curve left by $\pi/2$ radians, revealing that $\cos t = \sin(\pi/2 + t)$.

Next it is shown that a sine wave is generated by recording changes in air pressure caused by a vibrating tuning fork. This provides an opportunity to introduce *frequency* (the number of vibrations per second) and *amplitude* (the maximum height of the curve above the axis), and to illustrate these concepts with tones produced by musical instruments. Combinations of tones are represented graphically by adding corresponding y coordinates.

The repetitive or periodic nature of the sine curve is emphasized next. Other periodic waves are shown, and the program mentions Fourier's remarkable discovery that every periodic wave is a combination of sine and cosine waves with appropriate amplitudes and frequencies.

Some historical background is given, and it is shown how sines and cosines occur in trigonometry as ratios of lengths of sides of right triangles. The law of cosines, the law of sines, and the addition formulas are mentioned briefly; they are developed in greater detail in *Sines and Cosines, Parts II* and *III.* The program closes with a visual montage of vibrating systems.

1.3.2 Comments on Visualization Techniques Used in *Sines and Cosines, Part I*

Review of Prerequisites. This segment reminds the viewer that in similar triangles lengths of corresponding sides have the same ratios. Glowing lines show the corresponding sides as the ratios appear on the screen. To focus attention on the object being discussed, the edges are drawn in a brighter color than the background and triangle interiors. Animation shows what happens to perimeters under scaling, then reviews the definition of π as the ratio of the circumference of a circle to its diameter.

Circular Motion and Sine Waves. This segment shows water wheels, wagon wheels, windmills, automobiles and trains to illustrate that repetitive circular motion is present in machines that have dominated our way of life from ancient times to modern times. As the narrator says "modern times" the screen shows an amusing clip from Charlie Chaplin's movie by the same name.

Animation shows how circular motion is described mathematically by a point moving around a circle of radius 1, and how radian measure of angles occurs naturally by measuring the distance around the circle in terms of the radius.

To introduce a sine curve, the video shows a bright spot on the rim of a locomotive wheel that is transformed by animation into a point moving on a circle. What is actually seen depends on the point of view. The motion appears circular if it is viewed from a direction perpendicular to the plane of the circle. As the plane rotates about a vertical diameter, the spot appears to move along an oval curve (an ellipse), and when the plane has rotated 90° the motion appears straight up and down, as though the spot was casting a shadow. The height of the shadow above or below a horizontal diameter is plotted against the angle of rotation of the wheel to produce a sine curve. (Figure 1.1.)

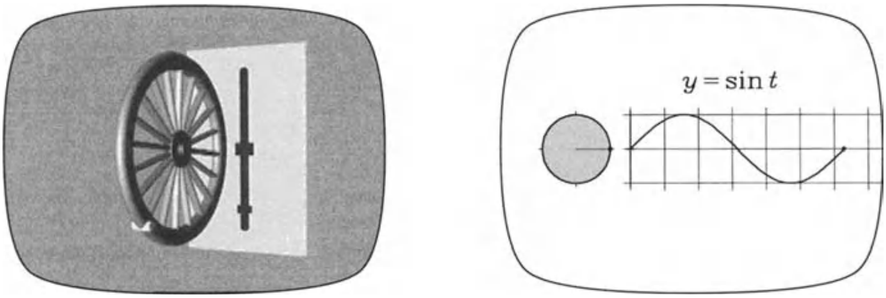

Fig. 1.1. Animation relating circular motion to a sine curve

Symmetry of Sine Waves. Symmetry properties of the circle give rise to special properties of sine curves. For example, as the angle t increases from 0 to $\pi/2$ radians, animation shows the graph of $\sin t$ increasing from 0 to 1. As the angle continues to increase from a right angle to a straight angle, the sine curve drops back to zero, forming a symmetric arch. The arch is symmetric about the line $t = \pi/2$ because the circle is symmetric about its vertical diameter. The symmetry of the arch implies that supplementary angles have the same sine; that is, the sine of an angle is equal to the sine of its supplement. Animation reveals this in an effective and convincing manner. (Figure 1.2.)

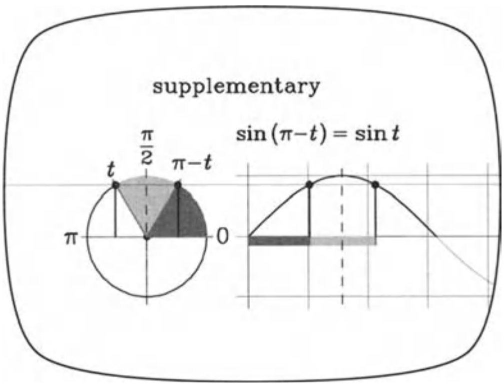

Fig. 1.2. Animation showing that the sine of an angle is equal to the sine of its supplement

Because the circle is also symmetric about its horizontal diameter, animation reveals that when t varies from π to 2π the sine is negative and its graph has the same shape as the first arch, except it is flipped over. The sine of $\pi + t$ is the negative of the sine of t.

The circle is also symmetric about the diameter $t = \pi/4$. When the sine curve is reflected about the vertical line $t = \pi/4$ it gives a new curve called the complementary sine, or cosine. Animation reveals that the cosine of $\pi/2 - t$ is equal to the sine of t. The angles $\pi/2 - t$ and t are called complementary angles and the sine of an angle is equal to the cosine of its complement.

Animation explains why the rectangular coordinates (x, y) of a point moving around a unit circle are given by $x = \cos t$ and $y = \sin t$.

It is then shown that the cosine curve can also be obtained by shifting the sine curve to the left by $\pi/2$ radians, revealing that the cosine of t is also equal to the sine of $\pi/2 + t$.

Sine Waves and Sound. This segment relates sine waves and musical tones. It displays the waveform for $\sin t$ and for an overtone $\sin 2t$, which has the same amplitude but twice the frequency. When the two tones are played simultaneously the resulting waveform is obtained by adding $\sin t + \sin 2t$, and animation shows how the graph of the sum is obtained by adding corresponding ordinates.

When a particular musical note is played on different instruments, such as a flute and a clarinet, the sound emitted will have a different quality because each instrument produces a different combination of overtones and amplitudes. Animation shows how the waveform of each instrument is obtained by combining waveforms with different frequencies and amplitudes.

Periodic Curves. When a point moves around a circle more than once, the angle of rotation t increases by 2π after each complete circuit. The shape of the graph of the sine is repeated, so that $\sin(t + 2\pi) = \sin t$. This provides an opportunity to introduce the general concept of periodicity. Different kinds of periodic functions are shown, such as square waves and saw-tooth waves. Then Fourier's remarkable theorem is demonstrated showing that any periodic wave is a linear combination of sine and cosine waves with varying frequencies and amplitudes. Animation gives a vivid description of the Gibbs phenomenon as it applies to a square wave.

Sines and Cosines as Ratios. A right triangle with legs a, b and hypotenuse c is transformed by scaling into a similar right triangle whose hypotenuse is the radius of a unit circle. Animation reveals that because of similarity the ratios a/c and b/c are equal, respectively, to the sine and cosine of one of the angles of the right triangle. The recap explains that later episodes will show how these ratios are used in trigonometry.

1.4 The Videotape *Sines and Cosines, Part II*

1.4.1 Brief Outline of the Program

The program begins with a brief review of *Sines and Cosines, Part I.* It explains that sines and cosines show up in many different contexts: As rectangular coordinates of a point moving on a unit circle; as waves generated by musical sounds and other periodic phenomena; and as ratios of sides of right triangles.

Sines and Cosines, Part II deals with trigonometry, which studies relations between sides and angles of triangles. One of the principal uses of trigonometry is to determine distances that are difficult or impossible to measure directly. Problems of this type occur in astronomy, large-scale construction, navigation, and surveying by triangulation. Two important tools for solving such problems are: (1) A generalization of the Pythagorean Theorem called *the law of cosines*, which relates the lengths of the three sides and one angle in any triangle; and (2) *the law of sines*, which states that in any triangle the length of any side divided by the sine of the opposite angle is constant.

One of the major triumphs of surveying by triangulation is the Survey of India, which took more than a century to complete. The program describes how this survey was done and how it led to the determination of the height of Mt. Everest. The program also outlines a brief history of surveying instruments, from the dioptra of ancient times to orbiting satellites of modern times.

1.4.2 Comments on Visualization Techniques Used in *Sines and Cosines, Part II*

Trigonometry. A triangle has six parts: three angles and three sides. This is indicated visually by showing a triangle with its angles shaded and labeled with capital letters A, B, C, and the opposite edges labeled with lower case letters a, b, c. The angles are displayed across the top of the screen with the corresponding edges as vertical line segments lying directly below the angles. (Figure 1.3.) Animation shows that the sum of the three angles is a straight angle, so if two angles are known we can find the third. Expanding or contracting a triangle does not change its angles, so if only the angles are known we cannot determine any of the sides. But if one side and two other parts are known, the remaining parts can be found using two properties of triangles that are the main focus of this program: *the law of cosines*, which explains how to find the length of one side of a triangle in terms of the lengths of the other two sides and the cosine of the angle between them, and *the law of sines*, which states that in any triangle the ratio of the length of a side to the sine of the opposite angle is constant.

The law of cosines is discussed first because it is a generalization of the Pythagorean Theorem, treated in an earlier program.

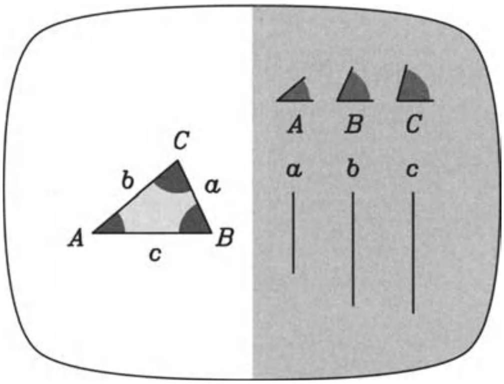

Fig. 1.3. Display of the six parts of a triangle

Sines, Cosines and the Pythagorean Theorem. To prepare the ground for an animated discovery of the law of cosines, there is a brief recap of one of the animated proofs of the Pythagorean Theorem from an earlier program. In a right triangle, oriented so that its hypotenuse is horizontal, squares are drawn adjacent to each of the three sides, with the area of each square being the square of the corresponding side of the triangle. The Pythagorean Theorem states that the area of the large square (adjacent to the hypotenuse) is the sum of the areas of the other two squares. If the legs of the right triangle are labeled a and b and the hypotenuse is labeled c, the Pythagorean Theorem states that $a^2 + b^2 = c^2$.

A perpendicular is dropped from the right angle to the hypotenuse and extended to divide the large square into two rectangles. Animation takes the square on one leg, colors its interior, and shears it into various parallelograms of equal area. One of these parallelograms is sheared again so it coincides with one of the rectangles in the large square. The same is done with the square on the other leg, thus completing the proof.

This animation shows how the essential ideas of a proof can be revealed without cluttering the diagram with complicated construction lines and labels. For example, the shear directions (parallel construction lines) are put on the screen as needed then immediately removed. Few professional mathematicians can reconstruct the details of Euclid's original proof, but almost everyone, professional or amateur, understands and remembers the animated proof.

The Law of Cosines. This section shows what happens in the foregoing proof of the Pythagorean Theorem when the same method is applied to a triangle that is not necessarily a right triangle. First the original right triangle with legs a and b and hypotenuse c is shown, with $c^2 = a^2 + b^2$. The hypotenuse c is kept fixed, and the vertex at the right angle is moved vertically up and down along a line perpendicular to the hypotenuse. When the vertex moves upward the sides a and b increase but c doesn't change, so $c^2 < a^2+b^2$. If the vertex moves down, then a and b decrease so $c^2 > a^2+b^2$. The inequalities are emphasized visually by showing the equation $c^2 = a^2 + b^2$ as being balanced about the equals sign. When the vertex angle is lifted to make it smaller than a right angle, the right member of the equation is lowered to suggest that $a^2 + b^2$ is heavier than c^2, and the equals sign changes to an inequality sign. Similarly, when the vertex angle is pushed down to make it larger than a right angle, the same process reveals that $c^2 > a^2+b^2$, and the equation tips the other way to suggest that in this case c^2 is heavier than $a^2 + b^2$.

To see these relations another way, we start again with the right triangle and divide the large square of area c^2 into two rectangles by dropping a perpendicular from the vertex angle (at the right angle) to the hypotenuse of length c. An animated hand lifts the vertex angle to make it smaller than a right angle, thus increasing each of a and b while keeping c fixed. The two rectangles filling the large square of area c^2 are sheared in reverse and we see that they do not completely fill the two squares of areas a^2 and b^2, revealing once more that $c^2 < a^2 + b^2$. And when the vertex angle is pushed down to make it larger than a right angle, the two sheared rectangles overlap the squares of areas a^2 and b^2, so $c^2 > a^2 + b^2$.

We return to the case in which the vertex angle is smaller than a right angle, with $c^2 < a^2 + b^2$, and calculate exactly how much the areas of the two rectangles differ from the areas a^2 and b^2. A simple calculation, using animation, reveals that the deficiency in area is the same for each rectangle and that each deficiency is exactly equal to the product $ab\cos C$, where C is the vertex angle. (Figure 1.4.) In this way we discover the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{1.1}$$

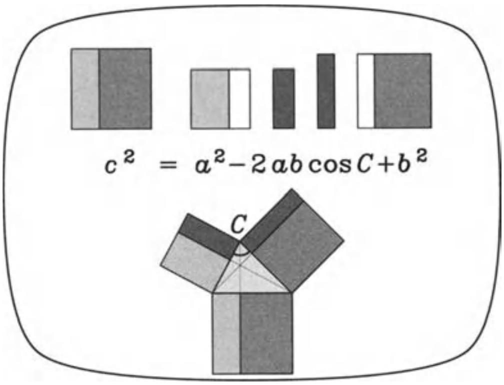

Fig. 1.4. Animation used to derive the law of cosines

When vertex angle C is greater than a right angle, $c^2 > a^2 + b^2$ and it is shown that the cosine law holds with $\cos C$ a negative number because C is greater than a right angle.

Applying the Law of Cosines. The law of cosines tells us how to find one side c of any triangle if we know the other two sides a and b and the angle C between them. There are corresponding versions of the cosine law for finding the other two sides:

$$a^2 = b^2 + c^2 - 2bc\cos A, \tag{1.2}$$

and

$$b^2 = a^2 + c^2 - 2ac\cos B, \tag{1.3}$$

where A is the angle opposite a, and B is the angle opposite b. The law of cosines involves four parts of a triangle, three sides and one angle. If three of these four parts are known we can use the law of cosines to find the fourth part.

The law of cosines can also be used to find the angles of a triangle when all three sides are known. In each of the above forms of the cosine law we can solve for the cosine of the angle in terms of the three sides. For example, $\cos C = (a^2 + b^2 - c^2)/(2ab)$. If the quotient is positive the angle C is smaller than a right angle. If the quotient is negative, angle C is greater than a right angle. And if the quotient is zero, then C is a right angle.

If we know two sides of a triangle, say a and b and the angle A opposite a, the law of cosines is a quadratic equation for c that can be solved by the quadratic formula. A quadratic equation can have two distinct real roots, so there may be two different triangles with given sides a, b and angle A.

If we know three angles and one side of a triangle, each form of the cosine law contains two unknown quantities and none of these equations alone suffices to find the other two sides. But in this case we can solve the problem by another tool called the law of sines.

The Law of Sines. The law of sines states that in any triangle the ratio of the length of a side to the sine of the opposite angle is the same for all three sides, so in a given triangle this ratio is constant. A proof is given by dropping a perpendicular from each vertex of a triangle to the opposite side and using the definition of the sine ratio three times. This shows that the three ratios $a/\sin A, b/\sin B$, and $c/\sin C$ are equal. In the workbook it is shown that this constant ratio is also equal to the diameter of the circle passing through the three vertices of the triangle. This is also proved using animation in *Sines and Cosines, Part III.*

Applying the Law of Sines. Examples are given to show how the law of sines can be used to find all the parts of a triangle if one side and two angles are known, a problem that cannot be solved using the law of cosines. The law of sines also gives a simple solution if two sides and one opposite angle is known, a case in which the law of cosines requires solving a quadratic equation.

Surveying by Triangulation. One of the most important applications of the law of sines is its use in surveying by triangulation from which accurate maps may be constructed. This method locates points called *stations* at the vertices of triangles. In surveying large geographic regions such as the continental United States or the subcontinent of India, thousands of triangles are joined together to form a triangulation system.

The lengths and directions of one or more *base lines* are determined by direct measurement and astronomical observations; and the lengths and directions of other lines in the triangles are calculated by measuring the angles between the various lines and applying trigonometric principles, usually the law of sines. The angles are measured with a precise instrument, such as a *theodolite*, which is part telescope and part protractor. Because a geodesic survey involves hundreds or sometimes thousands of triangles, measurements of angles and base line distances must be made with great care. To improve accuracy, overlapping triangles are used to form a belt of quadrilaterals. The video uses live action combined with animation to show how the process of triangulation takes place in actual practice.

The program also gives a brief history of surveying and surveying instruments, starting from primitive methods used by ancient Egyptians. It describes the dioptra, an instrument of early Greek origin used for leveling and measuring right angles, as well as later improvements that evolved after the development of the telescope and the magnetic compass. Video technology makes it possible to describe this history with many engaging images of instruments that are not readily accessible to most schoolteachers. The presentation of such history also reveals that mathematics plays an important role in human endeavor.

The Great Trigonometric Survey of India. This segment outlines the highlights of the Survey of India, which took more than a century to complete. The survey party also calculated the height of Mt. Everest, the highest mountain on earth. Computer animation shows how the law of sines was used to make six separate calculations from six different locations on the Indian plain below the summit. (Figure 1.5.) The average of the six calculations placed the height of the summit at 29,002 feet. The mountain was named in honor of George Everest, who played a major role in the Survey of India. Later calculations using satellites and electronic technology determined the

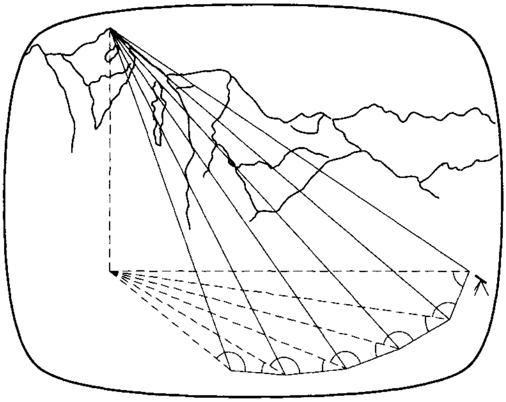

Fig. 1.5. The law of sines used to determine the height of Mt. Everest

height to be 29,028 feet, only 26 feet higher than the original calculations made a century earlier.

This segment also mentions some of the efforts made by expeditions that attempted to climb to the summit of Mt. Everest. Stock footage is shown of early expeditions that failed, and of the dramatic victorious assault by Edmund P. Hillary and Tenzing Norgay in 1953.

1.5 The Videotape *Sines and Cosines, Part III*

1.5.1 Brief Outline of the Program

After a brief review of *Sines and Cosines, Parts I and II*, this program relates the sine and cosine of an angle with the lengths of chords on a circle. This leads to several different derivations of the addition formulas for the sine and cosine of a sum of two angles. One of these is based on a famous theorem of Ptolemy on quadrilaterals inscribed in a circle. Several applications are given of the addition formulas. One shows that a combination of a sine wave with a cosine wave of the same frequency is another sine wave, possibly shifted. Another shows how the addition formulas make it possible to determine exact expressions for sines and cosines of many angles in terms of square roots of integers.

1.5.2 Comments on Visualization Techniques Used in *Sines and Cosines, Part III*

Sines and Chord Lengths. The origins of trigonometry go back to ancient astronomy. Early astronomers were interested in chords of circles because they thought planets moved in circular orbits. Around 150 B.C. the Greek mathematician Hipparchus of Rhodes, often called the father of trigonometry, wrote a treatise on chords of circles. This work is now lost, but many of the ideas survived in Claudius Ptolemy's *Almagest*, written some 300 years later in Alexandria. The video shows images of relevant historical documents: a Babylonian clay tablet containing a list of chords of circles; the title page of the *Almagest*; and a page of the *Almagest* that contains a table of chords of circles. In modern terminology, this would be equivalent to a table of sines.

The program explores the relation between sines and chord lengths. It starts with a circle and displays a central angle formed by two radii from the center. This angle cuts off an arc whose length depends on the size of the angle. The same arc is cut off by an inscribed angle whose vertex is on the circle. Animation shows why the measure of the central angle is twice that of the inscribed angle cutting off the same arc.

Many interesting consequences flow from this relation. The video shows a central angle and an inscribed angle cutting off the same arc. The central angle is kept fixed and the vertex of the inscribed angle is moved to different positions along the circle. No matter where the vertex lies on the circle, the measure of the inscribed angle is always equal to half the central angle. In other words, any two inscribed angles on a circle that cut off the same arc (and hence the same chord) must be equal in measure. Animation makes these properties easy to comprehend.

In particular, if the central angle is a straight angle the chord cut off is a diameter, so any inscribed angle cutting off a diameter is a right angle. This shows that any triangle inscribed in a semicircle, with one side along a diameter, is a right triangle with the diameter as hypotenuse.

This leads to an alternative proof of the law of sines. A triangle with three acute angles A, B, C is inscribed in a circle of diameter d. The respective sides opposite the angles have lengths a, b, c. As the vertex of angle A moves along the circle so that it always cuts off the same arc (hence the same chord) the measure of A does not change. The vertices B and C are kept fixed and the vertex of A moves to the point A' diametrically opposite the vertex C. (This is possible because A is acute.) Triangle ABC is a right triangle with a right angle at B and hypotenuse along the diameter of length d. The angles A and A' have equal measure. Therefore $\sin A = \sin A' = a/d$, the ratio of the opposite side to the hypotenuse. This is rewritten as $a/\sin A = d$. The argument is repeated by interchanging the roles of A, B and C to show that each of the other two ratios $b/\sin B$ and $c/\sin C$ is also equal to d. This gives the law of sines and also shows that the constant ratio is the diameter d of the circle through the three vertices of the triangle. (Figure 1.6.)

In particular, when $d = 1$ we see that $a = \sin A, b = \sin B$, and $c = \sin C$. In other words, in a circle of unit diameter the length of the chord cut off by an inscribed angle is equal to the sine of the inscribed angle. Consequently, Ptolemy's theorems on chords translate directly into theorems on sines. One of these is the addition formula described in the next segment.