Geometry, Mechanics, and Dynamics

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To Jerry Marsden
on the occasion of his 60th birthday,
with admiration, affection, and best wishes
for many more years of creativity.

Photo by Robert J. Paz

Contents

Preface

Jerry Marsden, one of the world's pre-eminent mechanicians and applied mathematicians, celebrated his 60th birthday in August 2002. The event was marked by a workshop on "*Geometry, Mechanics, and Dynamics*" at the Fields Institute for Research in the Mathematical Sciences, of which he was the founding Director. Rather than merely produce a conventional proceedings, with relatively brief accounts of research and technical advances presented at the meeting, we wished to acknowledge Jerry's influence as a teacher, a propagator of new ideas, and a mentor of young talent. Consequently, starting in 1999, we sought to collect articles that might be used as entry points by students interested in fields that have been shaped by Jerry's work. At the same time we hoped to give experts engrossed in their own technical niches an indication of the wonderful breadth and depth of their subjects as a whole.

This book is an outcome of the efforts of those who accepted our invitations to contribute. It presents both survey and research articles in the several fields that represent the main themes of Jerry's work, including elasticity and analysis, fluid mechanics, dynamical systems theory, geometric mechanics, geometric control theory, and relativity and quantum mechanics. The common thread running through this broad tapestry is the use of geometric methods that serve to unify diverse disciplines and bring a wide variety of scientists and mathematicians together, speaking a language which enhances dialogue and encourages cross-fertilization. We hope that this book will serve as a guide to these exciting and rapidly evolving areas, and that it will be a resource both for the student intent on contributing to one of these fields and to the seasoned practitioner who seeks a broader view.

Jerry is a unique figure in mathematical circles because his work has significantly influenced four often (alas!) separate research communities: pure mathematicians, applied mathematicians, physicists, and engineers. *Foundations of Mechanics* (with Ralph Abraham [294]), first published in 1967 while Jerry was a graduate student at Princeton, has for the past 35 years been a landmark and inspiration in the field of mechanics; during that time, Jerry and his collaborators have done extraordinary work in a huge variety of sub-fields of mechanics, geometry, and dynamics. Ralph Abraham recalls: "The first edition of *Foundations of Mechanics* included, in my Preface, a few words on the genesis of the book as Jerry's notes of my lectures in early 1966. I well recall the first meeting of that graduate course. At the outset I announced a desire for volunteers to make notes which might be duplicated for the use of students, as there was at that time no text we could follow. And at the end of that first meeting, only one volunteer: Jerry. He was a new face for me, and seemed rather young and quiet, and I told him I hoped that others would volunteer for a team effort on the notes. Well, there were no other volunteers, which was just as well. For shortly after each lecture Jerry would deliver a thick sheaf of handwritten notes, usually without a single error. Many details omitted in my talks were filled in with proofs, references, and so on, in the now-famous Marsden style. And the rest, as they say, is history. By now many people know that Jerry is an ideal coworker and coauthor, and I was lucky to be an early benefactor of his wonderful talents and personality."

A talented and prolific expositor, Jerry has written numerous other books, from elementary to advanced level, in addition to his many research articles. *Mathematical Foundations of Elasticity* (with Tom Hughes [300]) introduced a generation of engineers with appetites for abstraction to a unified and global approach to the subject, and his recent book *Introduction to Mechanics and Symmetry*, (with Tudor Ratiu [303]) has been remarkably useful to a wide range of scientists and engineers. When Jerry won the 1990 Norbert Wiener Prize (jointly with Michael Aizenman), he noted in his response to the citation that Wiener was "classifiable neither as a pure nor an applied mathematician. He had breadth and depth that worked together in a mutually supportive way." The same is true of Jerry: it is no accident that he began his career in mathematical physics, moved to a mathematics department, and is now working in the Division of Engineering and Applied Science at Caltech.

Jerry's influence on mathematical education has also been significant. His books on calculus and complex variables are widely used and, with their skillful blend of concreteness and abstraction, have influenced generations of undergraduates. Thorough and wide-ranging in their coverage, they leave the conscientious student with a solid grounding in both theoretical techniques and physical intuition. Jerry's Ph.D. and postdoctoral students, some of them now leaders in their fields, have made significant contributions in many areas themselves. In addition, Jerry has worked tirelessly for the mathematical community, serving on editorial boards and arranging conferences and workshops, all the while teaching a stellar array of undergraduate and graduate students and post-docs, first at UC Berkeley, and now at Caltech.

His extraordinarily influential paper with David Ebin [13], on the analysis of ideal fluid flows remains a classic in the field. It followed upon Arnold's 1966 paper[1] on ideal fluid flows, which showed how the Euler dynamics for

[1] Arnold, V. I. [1966], Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier, Grenoble, **16**, 319–361.

rigid bodies and fluids could be viewed as geodesic flow on SO(3) with a left-invariant metric, and on $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ — the volume preserving diffeomorphism group of a region Ω in $\mathbb{R}^3$ — with the right invariant metric defined by the fluid kinetic energy. Ebin and Marsden [13] put this work in the context of Sobolev (H^s) manifolds and showed that Arnold's geodesic flow on $H^s - \mathrm{Diff}_{\mathrm{vol}}(\Omega)$, the volume preserving diffeomorphisms of Ω to itself of Sobolev class H^s, comes from a smooth geodesic spray. This allowed them to show that the initial value problem for the Euler equations could be solved using Picard iteration and techniques from ordinary differential equation theory. Stephen Smale has remarked: "Jerry Marsden has many fine achievements to his credit. But I am particularly fond of his early work with David Ebin on the equations of fluid mechanics. There are many sides to this study. It gave formal ideas of Arnold great substance and provided an elegant way of presenting old and new fundamental work on the existence of solutions of Navier–Stokes and Euler equations. The rigorous group setting and one of the first important uses of infinite dimensional manifolds are there as well. Quite a milestone in mathematics!"

His body of work on "reduction theory," begun with Alan Weinstein, was an outgrowth of ideas developed by Smale (following Jacobi and others), who introduced the use of symmetry ideas in the context of tangent and cotangent bundles of configuration spaces with Hamiltonians in the form of kinetic plus potential energy. The Marsden and Weinstein paper [30] unified approaches of both Smale[2] and Arnold by putting this "reduction theory" in the context of symplectic manifolds. For instance, in the related Poisson context (developed by his student Richard Montgomery) if one starts with a cotangent bundle T^*Q and a Lie group G acting on Q, then the quotient $(T^*Q)/G$ is a bundle over $T^*(Q/G)$ with fiber $\mathfrak{g}^*$, the dual of the Lie algebra of G. As described by Marsden [170], "Thus, one can say — perhaps with only a slight danger of oversimplification — that reduction theory synthesizes the work of Smale, Arnold (and their predecessors of course) into a bundle, with Smale as the base and Arnold as the fiber." Reduction theory has now been used successfully in a wide variety of fields, and we refer the reader to the overview articles by Marsden [170; 227] as well as many of the articles in this volume for current applications.

From these works emerge more than specific theorems and techniques, deep and elegant as they may be. Viewing Jerry Marsden's contributions as a whole, one finds a clear, pedagogical, and fundamental approach to the subject of mechanics that blends geometry, analysis, and dynamics in powerful, yet practical ways. Thus, while developing abstract techniques in dynamical systems theory, Jerry also helped understand specific orbit trajectories (with a group at the Jet Propulsion Lab) that were used in the Genesis Discovery Mission, launched on August 8, 2001, [244]. In

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[2] Smale, S. [1970], Topology and mechanics, Invent. Math., **10**, 305–331; **11**, 45–64.

the course of developing the averaged fluid equations (with Holm, Ratiu, and Shkoller), he also contributed to their use in turbulent flow computations [267]. While working out infinite dimensional versions of the Melnikov method and Smale–Birkhoff theory to prove the existence of "Smale horseshoes" in the context of partial differential equations, the chaotic oscillations of a forced beam were being analyzed [71]; and while developing symplectic-energy-momentum preserving variational integrators based on discrete variational principles, Jerry contributed to specific projects (with Michael Ortiz) to simulate the crushing of aluminum cans and analyze fracture mechanics and collision problems [253; 284].

In each of the general areas noted above, we have solicited survey and research articles that illustrate more specifically how some of the methods pioneered by Marsden are currently being used.

For **Elasticity and Analysis**, the paper by J. M. Ball entitled "*Some open problems in elasticity*" is a self-contained overview which highlights some general open problems in elasticity theory, including some new results showing that local minimizers of the total elastic energy satisfy a weak form of the equilibrium equations. This is followed by the article of A. Mielke, "*Finite elastoplasticity, Lie groups and geodesics on* $\mathrm{SL}(d)$" which interprets notions of nonlinear plasticity theory in terms of Lie groups, among other things. The contribution of A. Lew and M. Ortiz, entitled "*Asynchronous variational integrators*" describes a new class of algorithms for nonlinear elastodynamics which is based upon a discrete version of Hamilton's principle.

D. D. Holm's article in **Fluid Mechanics**, "*Euler–Poincaré dynamics of perfect complex fluids*," describes the use of Lagrangian reduction by stages to derive the Euler–Poincaré equations for non-dissipative motion of exotic fluids such as liquid crystals, superfluids, Yang-Mills magnetofluids and spin-glass systems. Inclusion of defects, such as vortices, in the order parameters is also treated. S. Shkoller's contribution, "*The Lagrangian averaged Euler* ($\mathrm{LAE}-\alpha$) *equations with free-slip or mixed boundary conditions*", presents a simple proof of well-posedness of the Euler-α equations with novel boundary conditions. E. Knobloch's and J. Vega's article, "*Nearly inviscid Faraday waves*", explores some of the consequences of introducing small viscosity in the study of surface-gravity-capillary waves excited by vertical vibration of a fluid layer. The contribution of T. J. R. Hughes and A. A. Oberai, "*The variational multiscale formulation of LES with applications to turbulent channel flows*", studies turbulent two-dimensional equilibrium and three-dimensional non-equilibrium channel flows using a variational multi-scale formulation of Large Eddy Simulation (LES).

In **Dynamical Systems Theory**, M. Golubitsky and I. Stewart address "*Patterns of oscillation in coupled cell systems*". The dynamics of coupled cell systems both in biological contexts (animal gaits) and physical contexts (coupled pendula/Josephson junctions) are described, with an emphasis on the use of symmetry ideas. In particular, the issue of how the modeling assumptions dictate the kinds of equilibria and periodic solutions is explored. This is followed by the paper of A. Chenciner, J. Gerver, R. Montgomery, and C. Simó, "*Simple choreographic motions of N bodies*: *A preliminary study*". They describe the existence of new periodic solutions to the N-body problem in which all N masses trace the same curve without colliding. J. Scheurle and S. Walcher's "*On normal form computations*" closes this section by reviewing computational procedures involved in transforming a vector field into a suitable normal form about a stationary point.

For **Geometric Mechanics**, the paper by J. P. Ortega and T. Ratiu entitled "*The optimal momentum map*" discusses the (dare we say) classical Marsden–Weinstein reduction procedure and the use of a new optimal momentum map which more efficiently encodes symmetry information of the underlying Hamiltonian system. V. Guillemin's and C. Zara's "*Combinatorial formulas for products of Thom classes*" obtains combinatorial descriptions of the equivariant Thom class dual to the Morse–Whitney stratification of compact Hamiltonian G-manifolds. The paper of R. Littlejohn and K. Mitchell, "*Gauge theory of small vibrations in polyatomic molecules*," considers molecular vibrations in the context of gauge theory and fiber bundle theory.

In **Geometric Control Theory**, the paper by A. Bloch and N. Leonard, entitled "*Symmetries, conservation laws, and control*" traces the role of Marsden's ideas on reduction and symmetries in the setting of nonlinear control theory. Specific applications to the dynamics of rigid spacecraft with a rotor and the dynamics of underwater vehicles are considered in detail.

Finally, for **Relativity and Quantum Mechanics**, A. E. Fischer and V. Moncrief's article entitled "*Conformal volume collapse of 3-manifolds and the reduced Einstein flow*" describes the Hamiltonian reduction of Einstein's equations of general relativity and the process of volume collapse. They prove that collapse occurs either along circular fibers, embedded tori, or completely to a point, but surprisingly, always with bounded curvature. This is followed by M. Gotay's contribution "*On quantizing semisimple basic algebras*" which examines whether there exists a consistent quantization of the coordinate ring of a basic coadjoint orbit of a semisimple Lie group.

We hope that this collection of articles gives the reader some appreciation of both the unity and diversity of the topics influenced by Jerry Marsden's approach to mechanics. But here we wish to do more than survey his mathematical and scientific contributions; we also want to celebrate Jerry as a colleague and friend. It therefore seems appropriate to conclude with some personal reminiscences.

Phil Holmes: I first met Jerry in the summer of 1976, at a conference on dynamical systems at Southampton University, organised by David Rand and Brian Griffiths. He joined Nancy Kopell, John Guckenheimer and Ken Cooke as one of four mathematicians from the USA invited to that meeting. I had completed my Ph.D. in Engineering (experimental studies of dispersive wave propagation in structures) at the Institute of Sound and Vibration a couple of years earlier, and had begun working on nonlinear vibration problems with David Rand. We had done some single and finite degree of freedom problems, and I wanted to begin looking at PDEs in structural mechanics from a dynamical systems perspective. I believe it was in late 1975 that someone told me Jerry was working on a book about bifurcations and dimension reduction for such problems. I wrote to ask for more information and back came a huge package, re-taped and tied with string by UK customs, containing a 500+ page photocopy of the typescript of "*The Hopf Bifurcation and its Applications*" by Marsden and McCracken [297]. In financially-constrained Britain I had never seen more than fifty pages of xerox copies (all copies were xerox copies in those days) at one time, without special permission. I started reading, and I'm still reading Jerry's papers and trying to catch up.

Jerry and I began corresponding. We met at the Southampton conference and I subsequently visited him in Berkeley during a hectic job-seeking tour of the USA in the Fall of 1976, and again during his visit to Heriot–Watt University in Edinburgh as a Carnegie Fellow in the spring of 1977. This resulted in our first joint paper [54], and was the beginning of a twenty-five year collaboration and friendship which I hope will last at least another twenty five. For me, one of the high points of this was our paper [71], in which we gave one of the first examples of a PDE with chaotic solutions (Smale horseshoes), via an infinite-dimensional extension of the Smale–Birkhoff theorem and Melnikov's method. (John Guckenheimer gave another at about the same time via center-manifold reduction of a reaction-diffusion equation at a codimension-two bifurcation point.) After I had settled in the USA at Cornell University, Jerry invited me to Berkeley for the Spring semester of 1981, during which we wrote a series of papers [73; 77; 82] extending Melnikov type analyses to multi-degree-of-freedom Hamiltonian systems (although not without leaving a few gaps in our proofs to be filled by others, in the time-honored tradition of Poincaré).

While we have not written joint papers in the last ten years, his work at the interface of mechanics and mathematics has remained an inspiration for my own, and we have met once or twice every year and had countless scientific, editorial, organizational, and mathematico-political discussions and collaborations. Jerry is a mainstay of the Journal of Nonlinear Science, which I now edit, and I'm proud to serve as an advisor to the Springer Applied Mathematical Sciences Series which Jerry edits with Larry Sirovich and Stu Antman. I was even prouder to nominate him for the AMS–SIAM

Norbert Wiener Prize in 1990, and to support his successful nominations to the Royal Society of Canada and the American Academy of Arts and Sciences.

But rather than these well-deserved honors, I especially wish to celebrate Jerry's continuing emphasis on mentoring and encouraging young people. Few people outside academia, and few Deans and Presidents within it, realise that a large part of research is actually teaching: teaching bright but sometimes erratically-educated graduate students the necessary background and methods, teaching colleagues and collaborators about new advances, and teaching oneself all the things that no one else did. Jerry is a master teacher: in his many textbooks at all levels, and in his conference presentations and lecture courses, always delivered with elegance, polish, and a little humor. (He is almost the only person I know who can put content into powerpoint — although he's also careful to explain that it's not actually powerpoint.)

At Berkeley and Caltech Jerry has had, and continues to have, a succession of wonderful Ph.D. students and postdocs, many of whom have gone on to propagate his grand project of geometrizing mechanics (their names appear elsewhere in this volume). He has been equally generous with his time with young visitors (for many of whom, including myself, he raised the funds to invite), with the students of others, and simply with people who write or approach him to ask questions at conferences and workshops. I'm happy that we've been able to include articles contributed by several such colleagues in this Festschrift. Certainly, Jerry's interest and involvement in the early struggles of a mechanic poorly trained in mathematics was enormously encouraging to me. In those far-off days, from misty England, he seemed to me a senior scientist: a Professor from distant and fabled Berkeley, David Lodge's Euphoric State U. Now that we are both almost seniors, he no longer seems *that* much older, but he still knows a lot more geometry and analysis, and I'm still taking notes in the second row and having trouble with the homework.

Paul Newton: Like many of us, I first met Jerry in print. As a Freshman at Harvard in 1977, I learned Stokes' theorem, Green's theorem and the divergence theorem from his (and Tromba's) beautiful "*Vector Calculus*." Those who are familiar with the first edition and who are aware of Jerry's fascination with weather patterns will suspect that his favorite aspect of the book must have been the cloud formations on its cover. Mine was the elegant formulation of these theorems in terms of differential forms, something I had never seen in high school! I remember using this work to such an extent (so much for my social life) that today it is held together only by being wedged between two volumes on my shelf.

Fast forward eight years to 1985. While a postdoc at Stanford, I participated (inconspicuously) in a seminar series on dynamics which Jerry, together with Lieberman, ran at Berkeley, and it was there that I first took note of what I now know to be his uncommon blend of open-mindedness and depth of thought, coupled with a generosity of spirit, demonstrated so vividly in his mentoring of young mathematicians.

But it was when we both ended up in Los Angeles at roughly the same time (1993 in my case) that we became friends. I was busy working out the details of how the geometric phase manifested itself in the context of vortex dynamics problems, and Jerry's encouragement and insights were invaluable in helping me move from an early interest in nonlinear dispersive wave models to more general issues in applied dynamical systems theory. Our interests overlapped again when I showed him some problems involving the motion of vortices on a sphere (with applications to weather patterns!). This led him (with Sergey Pekarsky) to begin applying nonlinear stability techniques to relative equilibrium configurations of vortices on a sphere.

As I read and re-read many of his books and papers, I seem to fall farther and farther behind. But occasionally I look up to where it all began, the punished copy of "*Vector Calculus*" on my bookshelf, and I wonder if I would have become a mathematician had my professor chosen Edwards and Penney instead of Marsden and Tromba.

Alan Weinstein: My work with Jerry has two facets: (1) symplectic reduction, Poisson geometry, and applications to stability of continuum mechanical systems; (2) calculus books.

Our original work on reduction, which is probably one of the two or three most influential papers I have written, was stimulated by our listening to lectures of Smale around 1970; Smale had developed the theory in the special case of lifted actions on cotangent bundles. I think that my interest in abstract symplectic geometry, combined with some interest in physics, meshed perfectly with Jerry's interest in relativity and applied Hamiltonian dynamics. (One should always mention in this context that symplectic reduction was discovered independently at about the same time by Ken Meyer, though I think it might be fair to say that he conceived of this construction in narrower terms than we did.)

About 10 years later, we were attending the "dynamics seminar" in the Berkeley physics department, where Allan Kaufman and Robert Littlejohn were studying recent work of John Greene and Phil Morrison on the Hamiltonian structure of equations in plasma physics. These authors had a Poisson structure for the Maxwell–Vlasov equations which they found by trial and error and for which they checked the Jacobi identity by hand; according to Morrison, this took them 4 months of work, mostly calculations. Jerry and I spent about 6 months developing the right application of reduction to this problem, after which we could derive the correct bracket in 4 minutes, with the Jacobi identity coming for free. Another payoff was that we discovered an error in the Morrison–Greene formula.

It was Jerry's interest in applications which kept us going through much of the 80's, applying these Poisson brackets to applications of Arnol'd's general method for analyzing the stability of continuum-mechanical motions. Much of this work was also done in collaboration with Darryl Holm and Tudor Ratiu, and I eventually dropped out of the "group", but the subject has continued to evolve in the hands of Jerry and his collaborators (now including notably Steve Shkoller), the latest developments being the study of "α-Euler equations" and wide applications of Lagrangian (as opposed to Hamiltonian) methods for the study of stability.

On the calculus side, I recall that our collaboration started with a discussion at the end of a tennis game. Jerry was in contact with a new publisher who wanted to do a calculus book, and we had some new ideas for calculus teaching, based on the use of bifurcation ideas to replace the early introduction of limits (which eventually appeared only in a spin-off book called *Calculus Unlimited*). We went through several publishers and many versions of the book, and I never had time to play tennis again. I think that Jerry kept it (and squash) up, though.

Acknowledgments: The editors owe a debt of gratitude to Wendy McKay and Ross Moore. Without their combined expertise in LaTeX and other matters, this volume might not have been presented to Jerry until his 70th birthday. We also wish to thank the editors and staff at Springer-Verlag, particularly Achi Dosanjh and Elizabeth Young, for making this book a reality.

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Part I

Elasticity and Analysis

1

Some Open Problems in Elasticity

John M. Ball

To Jerry Marsden on the occasion of his 60th birthday

ABSTRACT Some outstanding open problems of nonlinear elasticity are described. The problems range from questions of existence, uniqueness, regularity and stability of solutions in statics and dynamics to issues such as the modelling of fracture and self-contact, the status of elasticity with respect to atomistic models, the understanding of microstructure induced by phase transformations, and the passage from three-dimensional elasticity to models of rods and shells. Refinements are presented of the author's earlier work Ball [1984a] on showing that local minimizers of the elastic energy satisfy certain weak forms of the equilibrium equations.

Contents

1 Introduction

In this paper I highlight some outstanding open problems in nonlinear (sometimes called finite) elasticity theory. While many of these will be well known to experts on analytic aspects of elasticity, I hope that the compilation will be of use both to those new to the field and to researchers in solid mechanics having different perspectives. Of course the selection of problems is a personal one, and indeed represents a list of those problems that I would most like to be able to solve, but cannot. In particular it concentrates on *general* open problems, or ones that illustrate general difficulties, rather than those related to very specific experimental situations, which is not to imply that the latter are not important or instructive. I have not included any open problems connected with the numerical computation of solutions, since I recently discussed some of these in Ball [2001].

The only new results of the paper are in connection with the problem of showing that local minimizers of the total elastic energy satisfy the weak form of the equilibrium equations. As I pointed out in Ball [1984a], there are hypotheses under which some forms of the equilibrium equations can be proved to hold, and in Section 2.4 I take the opportunity to present some refinements of this old work.

The paper is essentially self-contained, and can be read by those having no knowledge of elasticity theory. For those seeking further background on the subject I have written a short introduction (Ball [1996]) to some of the issues, intended for research students, which I hope is a quick and easy read. For more serious study in the spirit of this paper, the reader is referred to the books of Antman [1995], Ciarlet [1988, 1997, 2000], Marsden and Hughes [1983] and Šilhavý [1997]. Other excellent but older books and survey articles are Antman [1983], Ericksen [1977b], Gurtin [1981] and Truesdell and Noll [1965]. Valuable additional perspectives can be found in the books of Green and Zerna [1968], Green and Adkins [1970], and Ogden [1984].

It is an honour to dedicate this article to Jerry Marsden, both as a friend and in recognition of his important contributions to elasticity, and thus to help celebrate his many talents as a mathematician, thinker and writer.

2 Elastostatics

2.1 The Stored-Energy Function and Equilibrium Solutions

Consider an elastic body which in a reference configuration occupies the bounded domain $\Omega \subset \mathbf{R}^3$. We suppose that Ω has a Lipschitz boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$, where $\partial\Omega_1, \partial\Omega_2$ are disjoint relatively open subsets of $\partial\Omega$ and N has two-dimensional Hausdorff measure $\mathcal{H}^2(N) = 0$ (i.e., N

has zero area). Deformations of the body are described by mappings

$$\mathbf{y} : \Omega \to \mathbf{R}^3 ,$$

where $\mathbf{y}(\mathbf{x}) = \big(y_1(\mathbf{x}), y_2(\mathbf{x}), y_3(\mathbf{x})\big)$ denotes the deformed position of the material point $\mathbf{x} = (x_1, x_2, x_3)$. We assume that $\mathbf{y}$ belongs to the Sobolev space $W^{1,1}(\Omega; \mathbf{R}^3)$, so that in particular the *deformation gradient* $D\mathbf{y}(\mathbf{x})$ is well defined for a.e. $\mathbf{x} \in \Omega$. For each such $\mathbf{x}$ we can identify $D\mathbf{y}(\mathbf{x})$ with the 3×3 matrix $(\partial y_i / \partial x_j)$.

We require the deformation $\mathbf{y}$ to satisfy the boundary condition

$$\mathbf{y}\big|_{\partial\Omega_1} = \bar{\mathbf{y}}(\,\cdot\,) , \tag{2.1}$$

where $\bar{\mathbf{y}} : \partial\Omega_1 \to \mathbf{R}^3$ is a given boundary displacement.

We suppose for simplicity that the body is homogeneous, i.e., the material response is the same at each point. In this case the total elastic energy corresponding to the deformation $\mathbf{y}$ is given by

$$I(\mathbf{y}) = \int_\Omega W\big(D\mathbf{y}(\mathbf{x})\big)\, \mathrm{d}\mathbf{x} , \tag{2.2}$$

where $W = W(\mathbf{A})$ is the stored-energy function of the material. We suppose that $W : M_+^{3\times 3} \to \mathbf{R}$ is C^1 and bounded below, so that without loss of generality $W \geq 0$. (Here and below, $M^{m\times n}$ denotes the space of real $m \times n$ matrices, and $M_+^{n\times n}$ denotes the space of those $\mathbf{A} \in M^{n\times n}$ with $\det \mathbf{A} > 0$.) The *Piola–Kirchhoff stress tensor* is given by

$$\mathbf{T}_R(\mathbf{A}) = D_{\mathbf{A}} W(\mathbf{A}) . \tag{2.3}$$

By formally computing

$$\frac{\mathrm{d}}{\mathrm{d}\tau} I(\mathbf{y} + \tau\varphi)\big|_{\tau=0} = 0 ,$$

we obtain the *weak form of the Euler–Lagrange equation* for I, that is

$$\int_\Omega D_{\mathbf{A}} W(D\mathbf{y}) \cdot D\varphi \, \mathrm{d}\mathbf{x} = 0 \tag{2.4}$$

for all smooth φ with $\varphi|_{\partial\Omega_1} = 0$. This can be shown (cf. Antman and Osborn [1979]) to be equivalent to the balance of forces on arbitary subbodies. If $\mathbf{y}$, $\partial\Omega_1$ and $\partial\Omega_2$ are sufficiently regular then (2.4) is equivalent to the pointwise form of the equilibrium equations

$$\operatorname{div} D_{\mathbf{A}} W(D\mathbf{y}) = 0 \qquad \text{in } \Omega , \tag{2.5}$$

together with the natural boundary condition of zero applied traction

$$D_{\mathbf{A}} W(D\mathbf{y})\mathbf{n} = 0 \qquad \text{on } \partial\Omega_2 , \tag{2.6}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes the unit outward normal to $\partial\Omega$ at $\mathbf{x}$. (More generally, we could have prescribed nonzero tractions of various types on $\partial\Omega_2$, as well as including the potential energy of body forces such as gravity in the expression for the energy (2.2), but for simplicity we have not done this, since the main difficulties we address are already present without these additions.)

To avoid interpenetration of matter, it is natural to require that $\mathbf{y} : \Omega \to \mathbf{R}^3$ be *invertible*. To try to ensure that deformations have this property, we suppose that

$$W(\mathbf{A}) \to \infty \qquad \text{as } \det \mathbf{A} \to 0^+ . \tag{2.7}$$

So as to also prevent orientation reversal we define $W(\mathbf{A}) = \infty$ if $\det \mathbf{A} \leq 0$. Then $W : M^{3\times 3} \to [0, \infty]$ is continuous. Clearly if $I(\mathbf{y}) < \infty$ then

$$\det D\mathbf{y}(\mathbf{x}) > 0 \qquad \text{for a.e. } \mathbf{x} \in \Omega . \tag{2.8}$$

Since $\mathbf{y}$ is not assumed to be C^1, (2.8) does not imply even local invertibility. For studies of local and global invertibility in the context of elasticity, or relevant to it, see Ball [1981], Bauman and Phillips [1994], Ciarlet and Nečas [1985], Fonseca and Gangbo [1995], Giaquinta, Modica and Souček [1994], Meisters and Olech [1963], Šverák [1988] and Weinstein [1985].

We assume that for any elastic material the stored-energy function W is *frame-indifferent*, i.e.,

$$W(\mathbf{RA}) = W(\mathbf{A}) \qquad \text{for all } \mathbf{R} \in \mathrm{SO}(3) , \quad \mathbf{A} \in M^{3\times 3} . \tag{2.9}$$

In addition, if the material has a nontrivial isotropy group $\mathcal{S}$, W satisfies the material symmetry condition

$$W(\mathbf{AQ}) = W(\mathbf{A}) \qquad \text{for all } \mathbf{Q} \in \mathcal{S} , \quad \mathbf{A} \in M^{3\times 3} .$$

The case $\mathcal{S} = \mathrm{SO}(3)$ corresponds to an *isotropic* material.

For *incompressible* materials the deformation $\mathbf{y}$ is required to satisfy the constraint

$$\det D\mathbf{y}(\mathbf{x}) = 1 \qquad \text{for a.e. } \mathbf{x} \in \Omega .$$

All of the problems and results contained in this article have corresponding incompressible versions, some of which we cite in the references. However, in general we do not state these explicitly.

2.2 Existence of Equilibrium Solutions

There are two traditional routes to proving the existence of equilibrium solutions. The first, pioneered by Stoppelli [1954, 1955] and described in the book of Valent [1988], is to use the implicit function theorem in a suitable Banach space X to prove the existence of an equilibrium solution close to a given one, when the data of the problem are slightly perturbed. In order to