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Probabilistic Behavior of Harmonic Functions

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To our families:

<table>
<tr><td>Rosa,
Nidia
and
Carisa</td><td>and</td><td>Donna,
Richard
and
Cecilia</td></tr>
</table>

Preface

At both the level of simple analogy, and at the level of deep underlying techniques, probability and harmonic analysis are intimately related. An abundance of literature already exists concerning this point. The purpose of this monograph is to elucidate several more instances of this relationship.

Much of what we present in Chapters 2–6 of this monograph is drawn from several papers of the authors as well as two joint papers of the authors and Ivo Klemeš, although some results here are new. We will attempt to unify all these results. We do this not only by eliminating the overlap in these works, but by improving and refining our techniques. However, it is our goal to present more than just theorems from all these sources. Additionally, we wish to explain the connections between theorems, historical precedents of the theorems, the related directions other authors have taken, and present conjectures and areas for further study. Of paramount importance is our objective to clarify the probabilistic ideas and techniques which are at the heart of most of the results herein. Naturally, the work of numerous other authors is involved in many ways: as historical precedent, as motivation for many of the results and as indispensable parts of the proofs, techniques and ideas. These works will be discussed in the appropriate places and copious references will be given.

Throughout, we will consider harmonic functions u defined on the upper half space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$. Most of what we do is also valid for caloric functions, that is, for functions on $\mathbb{R}^{n+1}_+$ which are solutions of the heat equation. However, our primary focus will be on harmonic functions and since most of the results on caloric functions involve minor modifications of the corresponding results for harmonic functions, they will usually be relegated to the status of remarks.

For $\alpha > 0$ we set $\Gamma_\alpha(x) = \{(s,t) \in \mathbb{R}^{n+1}_+ : |x - s| < \alpha t\}$, which is a cone with vertex at $(x,0)$ and vertical axis $\{(x,t) : t > 0\}$, and for u harmonic on $\mathbb{R}^{n+1}_+$ we set

$$N_\alpha u(x) = \sup\{|u(s,t)| : (s,t) \in \Gamma_\alpha(x)\}$$

and

$$A_\alpha u(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(s,t)|^2 t^{1-n} ds dt \right)^{\frac{1}{2}} .$$

These are the classical nontangential maximal function of u and the Lusin area function of u, respectively. If u is the Poisson extension of a function f on $\mathbb{R}^n$, we will often write $N_\alpha f$ and $A_\alpha f$ instead of $N_\alpha u$ and $A_\alpha u$. The corresponding nontangential maximal function and Lusin area function for caloric functions are a slight variant of these and will be defined at an appropriate point later.

Let X_t be a continuous martingale starting at 0 and set $X_t^* = \sup_{0<s<t} |X_s|$, $X^* = \sup_{t>0} |X_t|$, $S_t(X) = \langle X \rangle_t^{1/2}$ and $S(X) = \langle X \rangle_\infty^{1/2}$, where $\langle X \rangle_t$ is the quadratic variation process of X_t at time t. We shall always assume that these martingales are with respect to the filtration of Brownian motion and that $X_0 = 0$. That X^* behaves analogously to $N_\alpha u$ and $S(X)$ behaves like $A_\alpha u$ is well appreciated and has been an important issue in probability and analysis, especially in the last 25 years. In this monograph, we continue in this direction.

In their ground-breaking paper [BG1], Burkholder and Gundy showed that the random variables X^* and $S(X)$ satisfy certain inequalities relating their distribution functions. These are now commonly called *good-λ inequalities.* (See also Burkholder [Bu1], [Bu3].) In the harmonic function setting the first good-λ inequalities were also proved by Burkholder and Gundy [BG2]. They were subsequently improved and refined by, among others, Burkholder [Bu4], Dahlberg [Da2], Fefferman, Gundy, Silverstein and Stein [FGSS], and Murai and Uchiyama [MU]. They were proved for caloric functions by A.P. Calderón and A. Torchinsky in [CT]. We shall also show such good-λ inequalities, but those we present here will be, in a sense to be made precise later, the sharpest possible attainable. These will involve what are known as "subgaussian" type estimates and in this regard are a perfect analogue of results first proved in the martingale setting. The advantage of these "subgaussian" estimates is that, just as in the case of martingales, they lead to *laws of the iterated logarithm* (LIL's) for harmonic functions. These LIL's will very precisely measure the relative growth of truncated versions of $N_\alpha u$ and $A_\alpha u$. In this way, the LIL's can be thought of as refinements of the classical theorem of Calderón and Stein which says that, up to sets of Lebesgue measure zero, the area function and the nontangential maximal function are finite or infinite on the same sets. This result also provides a harmonic analysis analogue of Kolmogorov's celebrated LIL for sequences of independent random variables as well as an analogue of Stout's [Sto] LIL for discrete martingales.

But what we present here will be more than just simple analogy. Our techniques will demonstrate the symbiosis enjoyed by harmonic analysis and probability. One of our good-λ inequalities, and one of our LIL's involve sharp estimates which control N by A. To prove these two results, we will actually reduce to the case of martingales. The strategy is a bit complicated and will be explained in detail in Chapter 2. The basic idea is to approximate $u(x, 2^{-n})$ by a dyadic martingale f_n with an error less than a truncated version of $A_\alpha u(x)$. This error is small enough so that results on martingales (in our case, good-λ inequalities and LIL's) can be transferred to the harmonic function setting. Our methods are a refinement of those introduced by Chang, Wilson and Wolff [CWW] to study the exponential square integrability of functions whose area function is bounded. These ideas are analogous to the *invariance principle* of Philipp and Stout [PS2]. Indeed, Philipp and Stout consider sequences of random variables $\{X_n\}$ (such as weakly dependent random variables, lacunary series, etc.) and set $U_t = \sum_{n<t} X_n$. They then show that under various conditions on X_n, the process $\{U_t : \ t \geq 0\}$

may be redefined, without changing its distribution, on a richer probability space together with standard Brownian motion $\{W_t : t \geq 0\}$ such that

$$U_t - W_t = 0(t^{1/2-\eta}) \tag{*}$$

almost surely as $t \longrightarrow \infty$, where $\eta > 0$ depends only on the given sequence $\{X_n\}$. From (*) one easily passes from theorems on Brownian motion, such as various types of LIL's, to the corresponding theorems for the sequence $\{X_n\}$. For example, since

$$\limsup_{t \longrightarrow \infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1 \tag{**}$$

almost surely, dividing both sides of (*) by $\sqrt{2t \log \log t}$ immediately gives a similar result for U_t. In fact, this does not even use the full strength of the error estimate $O(t^{1/2-\eta})$. In the same way, Stout's [Sto] LIL for martingales, from which we will obtain the upper half part of our LIL for harmonic functions, follows from a weaker version of (*) in Philipp and Stout [PS1] (Theorem 6.1.4 in Chapter 6 below). Thus, in this sense our LIL for harmonic functions, Theorem 3.0.4 below, is really reduced to the LIL for Brownian motion: the harmonic function is approximated by martingales which are in turn approximated by W_t. We direct the interested reader to Hall and Heyde [HH] for an extensive presentation of invariance principle techniques, as well as an abundance of references concerning these.

Our other good-λ inequality, and a Chung-type LIL for the nontangential maximal function and Lusin area function will involve sharp estimates which control A by N. These will be obtained in a more straightforward way, that is, we will not need to reduce to the case of martingales; yet, we will rely heavily on the probabilistic techniques. Our strategy will be to first prove good-λ inequalities in unbounded Lipschitz domains, that is, domains D of the form $D = \{(x, y) \in \mathbb{R}^{n+1} : y > \phi(x)\}$, where $\phi(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Lipschitz function; these turn out to be a natural analogue of stopping times in probability. The point we wish to stress here is that, as we will show in Section 4.2, once the Lipschitz domain inequalities are proved, one can literally copy line by line the martingale proofs of the LIL's over to the harmonic functions setting. These applications are *"la raison d'être"* for our study of Lipschitz domains in this monograph.

In addition to X^* and $S(X)$, there is a third random variable associated to a continuous martingale X_t. To define this, consider the quadratic variation process $\langle X \rangle_t$ of X and define a measure μ on $\mathbb{R}$ by $\mu(E) = d\langle X \rangle_t(\{t : X_t \in E\})$ where $d\langle X \rangle_t$ is the Riemann-Stieltjes measure on $[0, \infty)$ associated to the nondecreasing function $\langle X \rangle_t$. Lévy [Lé1] showed that μ is absolutely continuous with respect to Lebesgue measure so that there exists a function $L(a)$, called the 'local time', so that $\mu(E) = \int_E L(a)da$ for every Borel set $E \subseteq \mathbb{R}$. Set $L^* = \sup\{L(a) : a \in \mathbb{R}\}$; the random variable L^* is called the maximal local time. Gundy [Gu1] proposed the following harmonic analysis analogue of L^*: Let u be harmonic in $\mathbb{R}^{n+1}_+$ and let $r \in \mathbb{R}$. Then $(u(s,t) - r)^+$ is subharmonic in $\mathbb{R}^{n+1}_+$ so its distributional Laplacian,

$\Delta(u(s,t)-r)^+$, is a positive measure on $\mathbb{R}^{n+1}_+$. We then set

$$D_\alpha u(x;r) = \int_{\Gamma_\alpha(x)} \Delta(u(s,t)-r)^+ t^{1-n} ds dt$$

and

$$D_\alpha u(x) = \sup\{D_\alpha u(x;r) : r \in \mathbb{R}\}.$$

It turns out that $D_\alpha u(x;r)$, called the density of the area integral, is a harmonic function analogue of $L(a)$ and $D_\alpha u(x)$, called the maximal density, is an analogue of L^*. This is not quite apparent from our short preceding discussion, but will become clear in Chapter 5 when we discuss $D_\alpha u(x)$ in detail and present a change of variables formula due to Gundy and Silverstein [GS]. This change of variables formula is the analogue of the "occupation times formula" (see Revuz and Yor, [RY, p. 209]) for the local time. The function $D_\alpha u(x;r)$ can also be defined, as in Gundy [Gu1], in terms of the conditional expectation of the local time of the martingale obtained by composing the harmonic function u with the Brownian motion in the upper half space. From this representation, one can already obtain many interesting boundedness properties of this functional. In Davis [Dav2], and independently, in Bass [Bas1], good-λ inequalities relating X^* and L^*, and $S(X)$ and L^* are proved. In this monograph, we will prove sharp versions of these for $N_\alpha u$ and $D_\alpha u$, and $A_\alpha u$ and $D_\alpha u$. Again, this will be deeper than simple analogy. The proofs utilize exactly the same ideas as our proofs of good-λ inequalities for $N_\alpha u$ and $A_\alpha u$. One of these can be reduced to the case of martingales and the others are proved using estimates on Lipschitz domains coupled with stopping time arguments. Again, this use of probabilistic ideas and techniques will allow us to obtain inequalities that are in some sense the sharpest attainable and the subgaussian nature of these estimates will again lead to LIL's, this time of the Kesten-type [Ke1].

Organization

We have organized the material as follows. In Chapter 1 we present much of the basic material on harmonic functions that we use in this monograph. This chapter can, and probably should, be skipped by those with a solid knowledge of the basic principles of harmonic functions, without affecting the reading of the rest of the monograph. In Chapter 2, we introduce the technique of approximation by martingales, that is, our "invariance principle" which approximates $u(x, 2^{-n})$ by a dyadic martingale f_n with error controlled by an appropriate version of $A_\alpha u(x)$. We state those results in some degree of generality because we believe that such techniques could be applied in various other settings. In Chapter 3, we prove the Kolmogorov type LIL for harmonic functions. Here we will first use the approximation of Chapter 2 to reduce the upper bound of the LIL to the corresponding upper bound in the martingale LIL of Stout. We then discuss the lower bound of the LIL. The proof of the lower bound follows the standard strategy employed in proofs of LIL's for weakly dependent sequences, in that it combines the upper bound with conditional Borel-Cantelli arguments. Unfortunately, the proof is quite technical. Despite these technicalities, we will strive to illuminate the similarities with Stout's result for martingales. In Chapter 4, we again use the approximation by martingales to show the good-λ inequality which controls N by A. We then show estimates on Lipschitz domains involving N and A and combine these with stopping time arguments to give the other good-λ inequality involving these functions. We conclude the chapter with several applications, including a Chung-type LIL for harmonic functions and several ratio inequalities which relate various quantities involving N and A. In Chapter 5, we use the same ideas to do all of this for $N_\alpha u$ and $D_\alpha u$, and $A_\alpha u$ and $D_\alpha u$. These results are then used to show a Kesten-type LIL for harmonic functions. The chapter concludes with an application of the techniques of the chapter to Brossard and Chevalier's characterization of $L \log L$ within H^1. In Chapter 6, we show how to apply our results and techniques to the study of lacunary series and Bloch functions. In particular, we relate the results contained in this monograph to the classical results concerning these. Scattered throughout, the reader will find several **open problems** and questions of interest.

Notation

We shall use $|\cdot|$ to denote Lebesgue measure. As is customary, $C, C_1, C_2, \ldots$ will denote constants which depend only on certain fixed parameters such as the dimension or the aperture of cones, but whose values may change from line to line. Likewise, $C(\alpha), C(\beta), C_\alpha, C_\beta$, $C_{\alpha,\beta,n,\gamma}, \ldots$, etc. will denote constants which depend on the parameters indicated but whose values may change from line to line. Finally, we caution the reader against drawing inferences as to the meanings of notation; the notation herein is somewhat inconsistent. For example, $A^1_\alpha u(x;t)$ represents a version of the Lusin area integral with the integration taken over a cone of aperture α which is truncated from below at height t with $0 < t < 1$ and from above at height 1. However, $D^1_\alpha u(x;r)$ will represent a version of the D-functional with the integration taken over a cone of aperture α truncated at height 1; here the "r" has a completely different meaning than the "t" in $A^1_\alpha u(x;t)$. We do this not to torment, vex, or confuse the reader but to keep our notation consistent with that which is already entrenched in the literature. As it is our hope that this present work will interest the reader in these topics, this choice was made to facilitate the reader's further investigations into the literature. Nevertheless, some standardization of notation has occurred. For example, the notations $A_\alpha u(x)$ and $A_\alpha(u)(x)$ both occur in the sources from which we have drawn this material; we have settled on $A_\alpha u(x)$ throughout. For convenience, we have included a notation index.

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It is a pleasure to express our sincere appreciation and thanks to J. Brossard, D.L. Burkholder, R.F. Gundy, W. Philipp, and T. Wolff for the many invaluable conversations on the topics of this monograph. We wish to thank Ivo Klemeš for our collaborations on LIL's (Theorems 3.0.4, 3.0.5 and 3.0.6). We thank Joan Verdera for suggesting that we write a short survey article on LIL's and for the encouragement when it became clear that we were a little beyond this point. We wish to express our deepest thanks to Betty Gick who so kindly and so efficiently typed the first draft of the manuscript and for her patience in answering our countless questions concerning TEX. We would also like to thank Sheree Walsh who typed a few parts of the first draft of the manuscript and who also endured, with patience, countless questions concerning TEX. We gratefully acknowledge the support of the National Science Foundation throughout the period when the research presented here was conducted and during the preparation of this monograph. Part of the monograph was written when the second author was visiting the Department of Mathematics, University College, Galway, Ireland. He enjoyed the use of their facilities and resources, and enjoyed their hospitality. To them: *Go raibh maith agat.*

Finally, we are particularly grateful to Richard Gundy for the many questions, problems and conjectures which he has so generously shared with us for the last twelve years. Much of the work presented in here evolved from our efforts to prove his conjecture: the Kolmogorov LIL for harmonic functions. This is the work presented in Chapter 3. His never ending energy and enthusiasm have been a constant source of encouragement and motivation for our work.

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May 1998

Contents

Chapter 1

Basic Ideas and Tools

In this chapter we will introduce the basic definitions, theorems, and some of the analytic techniques and tools that will be used throughout the book. Here we will consider harmonic functions and their basic properties, the Poisson kernel, the Dirichlet problem, H^p spaces, and singular integrals. We will then state and prove some of the classical results relating the nontangential maximal function and Lusin area function. Our goal is not to give a comprehensive introduction to these topics, but rather to introduce, as quickly and efficiently as possible, the requisite background, both mathematical and historical, for what follows in the subsequent chapters. We provide proofs for most of the results in this chapter, especially those concerning harmonic functions on half spaces, since these occupy center stage throughout this monograph. However, it will be impossible, in the space of an introductory chapter, to present complete proofs of all the material mentioned above. Although we will be thorough with the development of harmonic functions, we will merely give references for some of the real analysis tools we will use, in particular, those results readily attainable in the literature. (We do assume that the reader is familiar with the rudiments of analysis.) Readers knowledgeable on these topics may skip this chapter, although it should serve such readers as a convenient reference. Those wishing a complete and comprehensive introduction to these topics are advised to consult some of the numerous texts already in existence: [ABR], [Du], [Fo], [Ga], [Ho], [Jou], [Koo], [St4], [St6], [SW2], [To] and [Zy2]. As we discussed in the Preface, and as the title clearly indicates, probabilistic ideas and techniques play an essential role in what we do in this monograph. Much of the material of this chapter can also be presented from this point of view. In order to maintain this chapter as short and as elementary as possible, we decided to present the analytic point of view in this introduction and refer the interested reader to [Dur] or [Bas2] for the probabilistic approach.

1.1 Harmonic functions and their basic properties

Definition 1.1.1 Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Suppose u is a twice continuously differentiable function on Ω, that is, all first and second derivatives of u exist and are continuous. We say u is harmonic on Ω if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \quad \text{on } \Omega.$$

Δ is called the Laplace operator or Laplacian.

Examples 1.1.2

(i) If $n = 1$ the definition implies that a harmonic function must be of the form $u(x) = ax + b,\ a, b \in \mathbb{R}$.

(ii) On $\mathbb{C}$ consider an analytic function $f(x,y) = u(x,y) + iv(x,y)$. The Cauchy-Riemann equations, $u_x = v_y,\ u_y = -v_x$ imply $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ and $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$, so that the real and imaginary parts of an analytic function are harmonic.

(iii) Suppose $n > 2$. Straightforward computation shows that $u(x) = |x|^{2-n}$ is harmonic on $\mathbb{R}^n - \{0\}$. $u(x)$ is called the fundamental solution of the Laplacian, or the potential kernel, or the Newtonian potential.

Theorem 1.1.3 (Mean value property for harmonic functions) *Suppose u is harmonic on $\Omega \subseteq \mathbb{R}^n$ and $B(a,r)$ is a ball which has $\overline{B(a,r)} \subseteq \Omega$. Then*

$$u(a) = \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u(y) d\sigma(y).$$

Here σ is surface measure on $\partial B(a,r)$.

Proof: We recall Green's theorem (also known as Green's identity): If W is a domain with piecewise C^1 boundary and if α and β are C^2 functions on an open set containing W, then

$$\int_W \alpha \Delta \beta - \beta \Delta \alpha dx = \int_{\partial W} \alpha \frac{\partial \beta}{\partial \nu} - \beta \frac{\partial \alpha}{\partial \nu} d\sigma.$$

Here dx denotes the usual Lebesgue measure on $\mathbb{R}^n$, $d\sigma$ is surface measure on ∂W, and $\frac{\partial}{\partial \nu}$ denotes differentiation in the direction of the outward normal to W.

We first suppose $n > 2$. Apply Green's theorem to the functions $u(x)$ and $|x-a|^{2-n}$ on $B(a,r) \backslash B(a,\varepsilon)$, where $\varepsilon < r$ is small. This yields:

$$\int_{B(a,r)\backslash B(a,\varepsilon)} u(x)\Delta(|x-a|^{2-n}) - |x-a|^{2-n}\Delta u(x) dx =$$
$$\int_{\partial(B(a,r)\backslash B(a,\varepsilon))} u(x) \frac{\partial}{\partial \nu} (|x-a|^{2-n}) - |x-a|^{2-n} \frac{\partial u}{\partial \nu} (x) d\sigma. \tag{1.1.1}$$

On $B(a,r)\backslash B(a,\varepsilon), u(x)$ is harmonic as is $|x-a|^{2-n}$ by example 1.1.2(iii). Thus the left hand side of (1.1.1) is zero. Also, $\frac{\partial}{\partial\nu}|x-a|^{2-n} = (2-n)|x-a|^{1-n}\frac{\partial}{\partial\nu}|x-a|$. On $\partial B(a,r)$, $\frac{\partial}{\partial\nu}|x-a| = 1$ and on $\partial B(a,\varepsilon)$, $\frac{\partial}{\partial\nu}|x-a| = -1$. Thus, on $\partial B(a,r)$, $\frac{\partial}{\partial\nu}|x-a|^{2-n} = (2-n)r^{1-n}$ and on $\partial B(a,\varepsilon)$, $\frac{\partial}{\partial\nu}|x-a|^{2-n} = -(2-n)\varepsilon^{1-n}$. These observations allow us to rewrite (1.1.1) as

$$\begin{aligned} 0 &= (2-n)r^{1-n}\int_{\partial B(a,r)} u d\sigma - (2-n)\varepsilon^{1-n}\int_{\partial B(a,\varepsilon)} u d\sigma \\ &- r^{2-n}\int_{\partial B(a,r)} \frac{\partial u}{\partial\nu}\, d\sigma - \varepsilon^{2-n}\int_{\partial B(a,\varepsilon)} \frac{\partial u}{\partial\nu}\, d\sigma. \end{aligned} \tag{1.1.2}$$

Apply Green's theorem to the functions u and 1 on $B(a,r)$. This yields:

$$\int_{B(a,r)} u\Delta 1 - 1\Delta u\, dx = \int_{\partial B(a,r)} u\frac{\partial 1}{\partial\nu} - 1\,\frac{\partial u}{\partial\nu}\, d\sigma.$$

This quickly reduces to $0 = \int_{\partial B(a,r)} \frac{\partial u}{\partial\nu}\, d\sigma$ and similarly, $\int_{\partial B(a,\varepsilon)} \frac{\partial u}{\partial\nu}\, d\sigma = 0$. Thus (1.1.2) reduces to

$$\frac{1}{\varepsilon^{n-1}}\int_{\partial B(a,\varepsilon)} u d\sigma = \frac{1}{r^{n-1}}\int_{\partial B(a,r)} u d\sigma. \tag{1.1.3}$$

Since $\sigma(\partial B(a,\varepsilon)) = \varepsilon^{n-1}\omega_{n-1}$, $\sigma(\partial B(a,r)) = r^{n-1}\omega_{n-1}$, where ω_{n-1} is a constant depending only on n (its precise value is inconsequential here; see Remark 1.2.2 below), (1.1.3) may be rewritten as

$$\frac{1}{\sigma(\partial B(a,\varepsilon))}\int_{\partial B(a,\varepsilon)} u d\sigma = \frac{1}{\sigma(\partial B(a,r))}\int_{\partial B(a,r)} u d\sigma.$$

Since u is continuous at a, we can let $\varepsilon \to 0$ in this last expression to obtain the conclusion of the theorem.

When $n = 2$, the result can be proved using the exact same argument with the function $\log|x|$ in place of $|x|^{2-n}$. Because $\log|x|$ is locally the real part of an analytic function, it is harmonic.

When $n = 1$, we have already observed that this implies u is linear. The result is clear in this case.

We remark that the Green's theorem techniques of this last proof and variations of this argument will be used repeatedly in this text – already again in the next section, and especially throughout Chapters 4 and 5. As a consequence of this last theorem we have:

Theorem 1.1.4 *Suppose $\phi \in L^1(\mathbb{R}^n)$ is radial, that is, $\phi(x) = f(|x|)$ for some $f\colon [0,\infty) \to \mathbb{R}$, and suppose $\int_{\mathbb{R}^n}\phi(x)dx = 1$. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and u harmonic on Ω. Then if $a \in \Omega$, and supp $\phi(\cdot - a) \subseteq \Omega$,*

$$u(a) = \int_{\mathbb{R}^n} \phi(x-a)u(x)dx.$$

Proof: Write the integral $\int_{\mathbb{R}^n} \phi(x-a)u(x)dx$ in polar coordinates and use Theorem 1.1.3.

Corollary 1.1.5 *Suppose u is harmonic on a domain $\Omega \subseteq \mathbb{R}^n$ and $B(a,r) \subseteq \Omega$. Then*

$$u(a) = \frac{1}{|B(a,r)|} \int_{B(a,r)} u(x)dx.$$

Proof: Take $\phi(x) = \chi_{B(0,r)}(x)\,/|B(0,r)|$ in Theorem 1.1.4.

Theorem 1.1.6 (The maximum principle for harmonic functions) *Let $\Omega \subseteq \mathbb{R}^n$ be a connected domain and let u be harmonic and real valued on Ω. Then if u attains a maximum in Ω, u must be constant in Ω.*

Proof: Suppose u attains a maximum at $a \in \Omega$. Choose $r > 0$ such that $\overline{B(a,r)} \subseteq \Omega$. Then since $u(a)$ is the average of u over $B(x,r)$ by Corollary 1.1.5, and u is continuous, we must have $u(x) \equiv u(a)$ on $B(a,r)$. Hence the set $\{x \in \Omega \colon u(x) = \max_{y\in\Omega} u(y)\}$ is open. By continuity, this set is also closed. Then by connectedness, this set is all of Ω.

Corollary 1.1.7 *If u is harmonic on a bounded connected domain Ω and continuous on $\overline{\Omega}$, then u attains its maximum on $\partial\Omega$.*

Proof: $\overline{\Omega}$ is compact so u attains its maximum somewhere on $\overline{\Omega}$. If the maximum is attained in the interior, the theorem and the continuity of u on $\overline{\Omega}$ imply u is constant on $\overline{\Omega}$ hence, in this case the maximum is also attained on $\partial\Omega$.

Remarks 1.1.8 A function u defined on a domain Ω is called subharmonic on Ω if it is upper semicontinuous and if for each $a \in \Omega$ there exists an $r_0 > 0$ (which depends on a), such that whenever $r < r_0$, then $\overline{B(a,r)} \subseteq \Omega$, and

$$u(a) \leq \frac{1}{|B(a,r)|} \int_{B(a,r)} u(x)dx. \tag{1.1.4}$$

We note that this was the only property of u used in the proof of Theorem 1.1.6. Consequently, the maximum principle remains valid for subharmonic functions.

If u is C^2 on a domain Ω, the Green's theorem argument of Theorem 1.1.3 can be adapted to show that if $\Delta u \geq 0$ on Ω, then

$$u(a) \leq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u\, d\sigma$$

whenever $\overline{B(a,r)} \subseteq \Omega$. Arguing as in Theorem 1.1.4 and Corollary 1.1.5 leads to the conclusion that then u satisfies (1.1.4). Consequently, if u is C^2 on a domain Ω and if $\Delta u \geq 0$ on Ω, then u is subharmonic on Ω.

A function u defined on a domain Ω is called superharmonic if $-u$ is subharmonic. Equivalently, we could require u to satisfy an inequality like (1.1.4), but with the $\leq$ there replaced by $\geq$. Because of this simple relationship between subharmonic and superharmonic functions, there is no need to discuss both. Accordingly, we will not mention superharmonic functions again.

1.2 The Poisson kernel and Dirichlet problem for the ball

The mean value property for harmonic functions essentially says that we can recover the value of a harmonic function at the center of a ball by taking its average over the surface of the ball. Our first goal in this section is to show that the value of a harmonic function at a point within a ball (not necessarily the center) can be obtained by taking a weighted average of the harmonic function over the surface of the ball.

Definition 1.2.1 Let B denote the unit ball in $\mathbb{R}^n$ and ω_{n-1} denote the surface area of ∂B. For $x \in B,\ y \in \partial B$ we set $P(x,y) = \frac{1}{\omega_{n-1}} \frac{1-|x|^2}{|x-y|^n}$. $P(x,y)$ is called the Poisson kernel for B.

Remark 1.2.2 $\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$; see [Fo] or [SW2] for this computation. With this notation, ω_n represents the surface area of the ball in $\mathbb{R}^{n+1}$, so that ω_n is the surface area of an n-dimensional manifold. Other authors let ω_n denote the surface area of the ball in $\mathbb{R}^n$. For us, the actual value of ω_n is irrelevant; what is important is that the constant in $P(x,y)$ is chosen so that Lemma 1.2.5(b) below holds.

Theorem 1.2.3 *Suppose u is harmonic on the unit ball B of $\mathbb{R}^n$ and continuous on $\overline{B}$. Then for $x \in B$,*

$$u(x) = \int_{\partial B} P(x,y)u(y)d\sigma(y).$$

Proof: The proof is similar to the proof of the mean value theorem for harmonic functions. In fact we note that if $x = 0$ the conclusion of the theorem is exactly the mean value theorem for harmonic functions. So we may suppose $x \neq 0$. We first consider the case $n > 2$. We fix $x \in B$ and apply Green's theorem to the functions $u(y)$ and

$$G(x,y) = |y-x|^{2-n} - |x|^{2-n}\left|y - \frac{x}{|x|^2}\right|^{2-n}$$

on the domain $B \backslash B(x,\varepsilon)$, where ε is chosen so that $B(x,2\varepsilon) \subseteq B$. Before doing this we first claim:

(i) $G(x,y) = 0$ if $|y| = 1$

(ii) $G(x,y)$ is harmonic on $B\backslash B(x,\varepsilon)$ (as a function of y).

To see i) we use the identity $\left|\frac{y}{|y|} - |y|x\right| = \left|\frac{x}{|x|} - |x|y\right|$, $x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,\ x \neq 0,\ y \neq 0$ which follows readily by squaring the expressions on each side of the equation. Thus, if $|y| = 1$ and $x \neq 0$,

$$\left|y - \frac{x}{|x|^2}\right||x| = \left||x|y - \frac{x}{|x|}\right| = \left|\frac{y}{|y|} - |y|x\right| = |y-x|.$$

So if $|y| = 1,\ x \neq 0$,

$$|y-x|^{2-n} = |x|^{2-n}\left|y - \frac{x}{|x|^2}\right|^{2-n},$$

which is i). To see ii) note that neither x nor $\frac{x}{|x|^2}$ is in the domain $B\backslash B(x,\varepsilon)$ so that ii) follows from example 1.1.2(iii). Now applying Green's theorem we obtain:

$$\begin{gathered}\int_{B\backslash B(x,\varepsilon)} G(x,y)\Delta_y u(y) - u(y)\Delta_y G(x,y)dy = \\ \int_{\partial(B\backslash B(x,\varepsilon))} G(x,y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial}{\partial \nu_y}G(x,y)d\sigma(y)\end{gathered} \tag{1.2.1}$$

Here we have written Δ_y and $\frac{\partial}{\partial \nu_y}$ to emphasize that derivatives are both taken with respect to y. The harmonicity of $u(y)$ and $G(x,y)$ on $B\backslash B(x,\varepsilon)$ immediately imply that the left hand side of (1.2.1) is 0. Also, if $y \in B,\ G(x,y) = 0$. Then (1.2.1) reduces to:

$$\begin{aligned}\int_{\partial B(x,\varepsilon)} G(x,y)\frac{\partial}{\partial \nu_y}u(y)d\sigma(y) &= \int_{\partial B} u(y)\frac{\partial G}{\partial \nu_y}(x,y)d\sigma(y) \\ &+ \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial G}{\partial \nu_y}(x,y)d\sigma(y).\end{aligned} \tag{1.2.2}$$

We want to let $\varepsilon \downarrow 0$ in (1.2.2). We note that since $\frac{x}{|x|^2} \notin B$, then $G(x,y) = 0(|y-x|^{2-n})$ for $y \in \partial B(x,\varepsilon)$ as $\varepsilon \downarrow 0$. Further note that $\frac{\partial}{\partial \nu_y}u(y)$ is bounded on a neighborhood of x. Thus, for ε small,

$$\begin{aligned}\left|\int_{\partial B(x,\varepsilon)} G(x,y)\ \frac{\partial}{\partial \nu_y}\ u(y)d\sigma(y)\right| &\leq C\int_{\partial B(x,\varepsilon)} |y-x|^{2-n}d\sigma(y) \\ &= C\varepsilon^{2-n}\int_{\partial B(x,\varepsilon)} d\sigma(y) \\ &= \varepsilon^{2-n}\varepsilon^{n-1} = C\varepsilon\end{aligned}$$

which tends to 0 as $\varepsilon \downarrow 0$. Also, for $y \in \partial B(x,\varepsilon)$,

$$\begin{aligned}\frac{\partial}{\partial \nu_y} G(x,y) =&(2-n)|y-x|^{1-n}\frac{\partial}{\partial \nu_y}|y-x| - \frac{\partial}{\partial \nu_y}\left(|x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{2-n}\right)\\ =&(2-n)\varepsilon^{1-n}(-1) - \frac{\partial}{\partial \nu_y}\left(|x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{2-n}\right).\end{aligned}$$

The second term is bounded for all y in a neighborhood of x. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} u(y)\,\frac{\partial G}{\partial \nu_y}(x,y)d\sigma(y) =& \lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} -(2-n)\varepsilon^{1-n}u d\sigma\\ &- \lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} u\,\frac{\partial}{\partial \nu_y}\left(|x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{2-n}\right)d\sigma(y).\end{aligned} \tag{1.2.3}$$

and the second term on the right hand side of (1.2.3) vanishes. Since u is continuous at x and $\sigma(B(x,\varepsilon)) = \omega_{n-1}\varepsilon^{n-1}$, (1.2.3) becomes

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} u(y)\,\frac{\partial G}{\partial \nu_y}(x,y)d\sigma(y) = -(2-n)\omega_{n-1}u(x).$$

Then, letting $\varepsilon \downarrow 0$ in (1.2.2) yields:

$$u(x) = \frac{1}{(2-n)\omega_{n-1}}\int_{\partial B} u(y)\,\frac{\partial G(x,y)}{\partial \nu_y}\,d\sigma(y) \tag{1.2.4}$$

The proof will be completed by showing that $\frac{1}{(2-n)}\frac{\partial G}{\partial \nu_y}(x,y) = \frac{1-|x|^2}{|x-y|^n}$. We have:

$$\begin{aligned}\frac{1}{2-n}\frac{\partial G}{\partial \nu_y}(x,y) &= \frac{1}{2-n}\,\frac{\partial}{\partial \nu_y}\left[|y-x|^{2-n} - |x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{2-n}\right]\\ &= |y-x|^{1-n}\,\frac{\partial}{\partial \nu_y}|y-x| - |x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{1-n}\frac{\partial}{\partial \nu_y}\left|y-\frac{x}{|x|^2}\right|\\ &= |y-x|^{1-n}\sum_{j=1}^{n} y_j\cdot\frac{\partial}{\partial y_j}(|y-x|)\\ &\quad - |x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{1-n}\sum_{j=1}^{n} y_j\cdot\frac{\partial}{\partial y_j}\left(\left|y-\frac{x}{|x|^2}\right|\right)\\ &= |y-x|^{1-n}\sum_{j=1}^{n} y_j\cdot\frac{(y_j-x_j)}{|y-x|} - |x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{1-n}\sum_{j=1}^{n} y_j\cdot\frac{y_j-\frac{x_j}{|x|^2}}{|y-\frac{x}{|x|^2}|}\\ &= \frac{|y|^2-x\cdot y}{|x-y|^n} - |x|^{2-n}\left|y-\frac{x}{|x|^2}\right|^{-n}\left(|y|^2-\frac{x\cdot y}{|x|^2}\right)\\ &= \frac{|y|^2-x\cdot y}{|x-y|^n} - \frac{|x|^2|y|^2}{|x-y|^n} + \frac{x\cdot y}{|x-y|^n} = \frac{1-|x|^2}{|x-y|^n},\end{aligned}$$