Springer Proceedings in Mathematics & Statistics

Volume 28

For further volumes: http://www.springer.com/series/10533

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of select contributions from workshops and conferences in all areas of current research in mathematics and statistics, including OR and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today. José A. Ferreira • Sílvia Barbeiro • Gonçalo Pena Mary F. Wheeler Editors

Modelling and Simulation in Fluid Dynamics in Porous Media



Editors José A. Ferreira CMUC Department of Mathematics University of Coimbra Coimbra, Portugal

Gonçalo Pena CMUC Department of Mathematics University of Coimbra Coimbra, Portugal Sílvia Barbeiro CMUC Department of Mathematics University of Coimbra Coimbra, Portugal

Mary F. Wheeler Inst. Computational Engineering and Sciences (ICES) University of Texas, Austin Austin, Texas, USA

ISSN 2194-1009 ISBN 978-1-4614-5054-2 DOI 10.1007/978-1-4614-5055-9 Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012951311

Mathematics Subject Classification (2010): 11R23, 11S40, 14H52, 14K22, 19B28

© Springer Science+Business Media New York 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

We are pleased to introduce the readers these proceedings containing a selection of papers from invited lectures and contributed talks presented at the Workshop on Fluid Dynamics in Porous Media that was held in Coimbra, Portugal, on September 12–14, 2011.

We believe that the Workshop on Fluid Dynamics in Porous Media was an occasion of inspiration for all participants and helpful for strengthening the links between researchers working in various modeling aspects in porous media.

This book includes research work of international recognized leaders in their respective fields and presents advances in both theory and applications. The contributions are devoted to mathematical modeling, numerical simulation, and their applications. These proceedings provide the readers an overview on the latest findings and new challenges in fluid dynamics in porous media, thus making them appealing to a multidisciplinary audience, including mathematicians, engineers, physicists, and computational scientists.

We express our gratitude to all the authors for their excellent contribution. We also wish to thank the generous collaboration of anonymous reviewers. This book could not have been successfully concluded without their assistance.

We gratefully acknowledge the financial support of UT Austin|Portugal Co-Lab, the Centre of Mathematics of University of Coimbra, Fundação para a Ciência e Tecnologia through European program COMPETE/FEDER, project UTAustin/MAT/0066/2008 "Reaction-Diffusion in Porous Media," and the Department of Mathematics of University of Coimbra. We also thank Springer for agreeing to publish this work, and in particular we express our appreciation for Meredith Rich who assisted us in the edition.

Coimbra, Portugal Coimbra, Portugal Coimbra, Portugal Austin, TX, USA José A. Ferreira Sílvia Barbeiro Gonçalo Pena Mary F. Wheeler

Contents

On the Coupling of Incompressible Stokes or Navier–Stokes and Darcy Flows Through Porous Media V. Girault, G. Kanschat, and B. Rivière	1
Comparison of Control Volume Analysis and Porous Media Averaging for Formulation of Porous Media Transport F. Civan	27
On the Energy Conservation Formulation for Flows in Porous Media Including Viscous Dissipation Effects V.A.F. Costa	55
Analytical and Numerical Study of Memory Formalisms in Diffusion Processes José A. Ferreira, E. Gudiño, and P. de Oliveira	67
Super-diffusive Transport Processes in Porous Media E. Sousa	87
Stochastic Forecasting of Algae Blooms in Lakes P. Wang, D.M. Tartakovsky, and A.M. Tartakovsky	99
Unfolding Method for the Homogenization of Bingham Flow R. Bunoiu, G. Cardone, and C. Perugia	109
An Integrated Capillary, Buoyancy, and Viscous-Driven Model for Brine/CO ₂ Relative Permeability in a Compositional and Parallel Reservoir Simulator	125
X. Kong, M. Delshad, and M.F. Wheeler	142
M. Patrício and R. Patrício	143

Implementing Lowest-Order Methods for Diffusive Problems with a DSEL JM. Gratien	157
Non-Darcian Effects on the Flow of Viscous Fluid in Partly Porous Configuration and Bounded by Heated Oscillating Plates S. Panda, M.R. Acharya, and A. Nayak	179
Experimental and Numerical Study of the Salt Dissolution in Porous Media F. Dorai, G. Debenest, H. Luo, H. Davarzani, R. Bouhlila, F. Laouafa, and M. Quintard	201

Contributors

M.R. Acharya OUAT, Bhubaneswar, India

R. Bouhlila Université de Tunis El-Manar, Tunis, Tunisia

R. Bunoiu LMAM, UMR 7122, Université de Lorraine et CNRS Ile du Saulcy, METZ Cedex 1, France

G. Cardone Department of Engineering, University of Sannio, Corso Garibaldi, Benevento, Italy

F. Civan Mewbourne School of Petroleum and Geological Engineering, The University of Oklahoma, Norman, OK, USA

V.A.F. Costa Departamento de Engenharia Mecânica, Universidade de Aveiro, Aveiro, Portugal

H. Davarzani Université de Toulouse; INPT, UPS; IMFT; Toulouse, France

G. Debenest Université de Toulouse; INPT, UPS; IMFT; Toulouse, France

M. Delshad Petroleum and Geosystems Engineering Department, The University of Texas at Austin, TX, USA

F. Dorai Université de Tunis El-Manar, Tunis, Tunisia

Université de Toulouse; INPT, UPS; IMFT; Toulouse, France

José A. Ferreira CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

V. Girault UPMC-Paris and CNRS, UMR, Paris Cedex 05, France

Department of Mathematics, Texas A&M University, TX, USA

J.-M. Gratien IFP Energies nouvelles, Rueil-Malmaison, France

E. Gudiño CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

G. Kanschat Department of Mathematics, Texas A&M University, TX, USA

X. Kong Petroleum and Geosystems Engineering Department, The University of Texas at Austin, TX, USA

F. Laouafa INERIS, Parc technologique ALATA, Verneuil-en-Halatte, France

H. Luo Université de Toulouse; INPT, UPS; IMFT; Toulouse, France

CNRS IMFT; Toulouse, France

IMFT-GEMP, Toulouse, France

A. Nayak Silicon Institute of Technology, Bhubaneswar, India

P. de Oliveira CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

S. Panda NIT Calicut, Calicut, India

M. Patrício CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

School of Technology and Management, Polytechnic Institute of Leiria, Leiria, Portugal

R. Patrício Active Space Technologies S.A., Coimbra, Portugal

C. Perugia DSBGA, University of Sannio, Benevento, Italy

M. Quintard Université de Toulouse; INPT, UPS; IMFT; Toulouse, France

B. Rivière Department of Computational and Applied Mathematics, Rice University, Houston, TX, USA

E. Sousa CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal

A.M. Tartakovsky Pacific Northwest National Laboratory, Richland, WA, USA

D.M. Tartakovsky Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA

P. Wang Pacific Northwest National Laboratory, Richland, WA, USA

M.F. Wheeler Center for Subsurface Modeling, The University of Texas at Austin, TX, USA

On the Coupling of Incompressible Stokes or Navier–Stokes and Darcy Flows Through Porous Media

V. Girault, G. Kanschat, and B. Rivière

Abstract In this chapter, we present the theoretical analysis of coupled incompressible Navier–Stokes (or Stokes) flows and Darcy flows with the Beavers–Joseph– Saffman interface condition. We discuss alternative interface and porous media models. We review some finite element methods used by several authors in this coupling and present numerical experiments.

1 Introduction

Mathematical and numerical modeling of coupled Navier–Stokes (or Stokes) and Darcy flows is a topic of growing interest. Applications include the environmental problem of groundwater contamination through rivers, the problem of flows through vuggy or fractured porous media, the industrial manufacturing of filters, and the biological modeling of the coupled circulatory system with the surrounding tissue. The most widely used coupling model is based on either the Beavers–Joseph or the simpler Beavers–Joseph–Saffman interface conditions. The Beavers–Joseph condition [9], which is a Navier-type slip with a friction

V. Girault (🖂)

G. Kanschat

B. Rivière Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005, USA e-mail: riviere@caam.rice.edu

UPMC-Paris 6 and CNRS, UMR 7598, F-75230 Paris Cedex 05, France

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA e-mail: girault@ann.jussieu.fr

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA e-mail: kanschat@tamu.edu

¹

condition involving the interaction between the tangential velocities at the interface, was derived experimentally in 1967. In 1971, it was simplified by Saffman [53] who, observing that usually the flow in the pores is negligible with respect to the free flow, replaced the difference in these two velocities by just the free flow velocity. In 2000, via homogenization arguments, the Beavers-Joseph-Saffman model was recovered by Jäger and Mikelić [36–38], Jäger et al. [39]. Since then the theoretical and numerical coupling of Stokes and Darcy flows has been addressed by many authors with a variety of settings ranging from a primal formulation in the Stokes region and either an H(div) formulation or a primal formulation in the Darcy region to a fully mixed formulation in both regions. Without being exhaustive, we refer to [5, 6, 12, 22–29, 34, 42, 44, 47, 48, 51, 55]. For instance, well-posedness of the coupled problem was established by Layton et al. in [44]; the authors used continuous finite elements in the Stokes region, H(div) elements in the Darcy region, and coupled both regions with a mortar. Rivière and Yotov in [51] and Gatica et al. in [29] proposed a primal formulation in the Stokes region coupled with a dual formulation in the Darcy region. Discacciati et al. proposed a primal formulation in both regions; see for example [24]. In [28], Gatica et al. analyzed a fully mixed formulation in both regions, introducing the deformation tensor in the Stokes subdomain. Finally, Arbogast and Brunson in [6] use a finite element formulation with continuity requirements changing between H^1 and H(div) as needed.

In contrast, there exists much less literature on the coupling of Navier–Stokes and Darcy flows. The readers can refer to [8, 15, 16, 32]. And finally, there exists some work on Stokes–Darcy flows coupled with the Beavers–Joseph interface condition. Albeit linear, this last problem is harder to formulate rigorously because the Darcy velocity lacks regularity at the interface; see the work of Cao et al. in [14].

Although this review focuses on the use of the Beavers–Joseph–Saffman condition to model coupled Navier–Stokes and Darcy flows, it also describes the approach of various authors in coupling Darcy or Brinkman and Stokes flows that can be easily extended to the nonlinear situation of the Navier–Stokes free flow.

2 Theoretical Analysis

2.1 Coupled Navier–Stokes and Darcy Systems

To simplify the discussion, we consider the three-dimensional problem; the twodimensional problem is treated in the same fashion.

Let Ω be a bounded, connected Lipschitz domain of \mathbb{R}^3 , with boundary $\partial \Omega$ and exterior unit normal vector \boldsymbol{n} , partitioned into two nonoverlapping regions: a porous region Ω_2 and a free fluid region Ω_1 , both assumed to be Lipschitz continuous

$$\Omega = \Omega_1 \cup \Omega_2.$$

Fig. 1 Problem setting

To simplify, we assume that each region is connected as in Fig. 1, but the analysis presented in this first part easily extends to regions with several connected components. Let $\Gamma_1 = \partial \Omega_1 \cap \partial \Omega$ denote the exterior boundary of the fluid region, $\Gamma_2 = \partial \Omega_2 \cap \partial \Omega$, the exterior boundary of the porous region, and $\Gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2$, the interface between the two regions. Since we are in \mathbb{R}^3 , we also assume that the surfaces Γ_1 , Γ_2 , and Γ_{12} have Lipschitz continuous boundaries.

In the fluid region Ω_1 , the constitutive equation for the Cauchy stress tensor **T** is

$$\boldsymbol{T}(\boldsymbol{u}_1, p_1) = 2\boldsymbol{\mu}\boldsymbol{D}(\boldsymbol{u}_1) - p_1\boldsymbol{I},\tag{1}$$

where \boldsymbol{u}_1 is the fluid velocity, $\boldsymbol{D}(\boldsymbol{u}_1) = \frac{1}{2} (\nabla \boldsymbol{u}_1 + \nabla \boldsymbol{u}_1^T)$ is the symmetric gradient or deformation tensor, p_1 is the fluid pressure, \boldsymbol{I} is the identity tensor, and $\mu > 0$ is the fluid viscosity. When substituted into the balance of linear momentum, after dividing by the constant density (keeping the same notation for the kinematic viscosity and pressure) and assuming that the flow has reached a steady state, we obtain the steady Navier–Stokes system

$$-\operatorname{div}\left(2\boldsymbol{\mu}\boldsymbol{D}(\boldsymbol{u}_{1})-p_{1}\boldsymbol{I}\right)+\boldsymbol{u}_{1}\cdot\nabla\boldsymbol{u}_{1}=\boldsymbol{f}_{1}\quad\text{in }\Omega_{1},$$
(2)

where f_1 is a density of fluid body forces. The conservation of mass and constant density give the incompressibility condition

$$\operatorname{div} \boldsymbol{u}_1 = 0 \quad \text{in } \Omega_1. \tag{3}$$

In the porous region Ω_2 , we assume that the fluid flow is laminar; we neglect the inertial effects in the fluid and only consider friction between the pores and the fluid. By neglecting also gravity, for simplicity, this gives the Darcy law:

$$\boldsymbol{u}_2 = -\boldsymbol{K}\nabla p_2, \text{ div } \boldsymbol{u}_2 = f_2 \quad \text{in } \Omega_2, \tag{4}$$

which in divergence form reads

$$-\operatorname{div}(\mathbf{K}\nabla p_2) = f_2 \quad \text{in } \Omega_2, \tag{5}$$



where u_2 is the fluid velocity, p_2 and the pore pressure, f_2 is a source or sink term, and **K** the permeability tensor divided by the viscosity, i.e.,

$$K=rac{\hat{K}}{\mu},$$

with \hat{K} the intrinsic permeability. We assume that K is bounded, symmetric, and uniformly definite. When the constant gravity g is included, the relation between the velocity and pressure is expressed by

$$\boldsymbol{u}_2 = -\boldsymbol{K}\nabla(p_2 - \rho gz),$$

where $\rho > 0$ is the constant density and z is the height.

For the interface equations, let n_{12} denote the unit normal to Γ_{12} pointing in Ω_2 and $\{t_{12}^1, t_{12}^2\}$ an orthonormal basis on the tangent plane to Γ_{12} . The incompressibility of the fluid implies continuity of the normal velocity :

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} = \boldsymbol{u}_2 \cdot \boldsymbol{n}_{12} = -\boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_{12}. \tag{6}$$

If Γ_{12} were a permeable boundary with no porous medium beyond, (6) could be complemented by $\mathbf{u}_1 \cdot \mathbf{t}_{12}^j = 0$, j = 1, 2. But at the interface between a fluid and a porous medium, we need conditions on the traction vector $\mathbf{T}\mathbf{n}$. The first condition is the balance of normal stresses:

$$p_2 = (\mathbf{T}\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = ((-2\mu \mathbf{D}(\mathbf{u}_1) + p_1 \mathbf{I})\mathbf{n}_{12}) \cdot \mathbf{n}_{12}.$$
(7)

For the second condition, Beavers and Joseph [9] postulated by experiment in 1967,

$$(\boldsymbol{u}_{1} - \boldsymbol{u}_{2}) \cdot \boldsymbol{t}_{12}^{j} = -G^{j}(\boldsymbol{T}\boldsymbol{n}_{12}) \cdot \boldsymbol{t}_{12}^{j} = -2\mu G^{j}(\boldsymbol{D}(\boldsymbol{u}_{1})\boldsymbol{n}_{12}) \cdot \boldsymbol{t}_{12}^{j}, \ j = 1, 2, \quad (8)$$

where

$$G^{j} = \frac{1}{\alpha} \sqrt{\frac{(\mathbf{K} \mathbf{t}_{12}^{j}, \mathbf{t}_{12}^{j})}{\mu}}, \ j = 1, 2,$$
(9)

and $\alpha > 0$ is a dimensionless constant depending on the structure of the porous medium. These are the Beavers–Joseph interface conditions. But Saffman [53], observing that u_2 is often negligible with respect to u_1 , proposed in 1971 to replace (8) by the simpler Navier-type condition:

$$\boldsymbol{u}_{1} \cdot \boldsymbol{t}_{12}^{j} = -2\mu G^{j} (\boldsymbol{D}(\boldsymbol{u}_{1})\boldsymbol{n}_{12}) \cdot \boldsymbol{t}_{12}^{j}, \ j = 1, 2.$$
(10)

These are the Beavers–Joseph–Saffman interface conditions; see also the references by Jäger and Mikelić [37, 38], for a derivation of these conditions by homogenization.

By eliminating the Darcy velocity and thus suppressing the index on u, we obtain the following system of equations:

$$\begin{cases} -2\mu \operatorname{div} \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p_1 = \boldsymbol{f}_1 \\ \operatorname{div} \boldsymbol{u} = 0 \end{cases} \quad \text{in } \Omega_1, \qquad (11)$$

$$-\operatorname{div}(\boldsymbol{K}\nabla p_2) = f_2 \quad \text{in } \Omega_2, \tag{12}$$

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n}_{12} = -\boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_{12} \\ -2\mu \sum_{j=1}^2 G^j (\boldsymbol{D}(\boldsymbol{u}) \boldsymbol{n}_{12}) \cdot \boldsymbol{t}_{12}^j = \sum_{j=1}^2 \boldsymbol{u} \cdot \boldsymbol{t}_{12}^j \\ ((-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I}) \boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} = p_2 \end{cases} \text{ on } \Gamma_{12}.$$
(13)

Since we are mainly interested in the coupling, we choose simple exterior boundary conditions; we split Γ_2 into two parts Γ_{2D} and Γ_{2N} , as in Fig. 1, and we prescribe for example:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1,$$

$$p_2 = 0 \quad \text{on } \Gamma_{2D},$$

$$(\boldsymbol{K} \nabla p_2) \cdot \boldsymbol{n}_2 = 0 \quad \text{on } \Gamma_{2N}.$$
(14)

Here we assume that $|\Gamma_{2D}| > 0$; otherwise, the source f_2 must satisfy the solvability condition:

$$\int_{\Omega_2} f_2 \, d\boldsymbol{x} = 0. \tag{15}$$

Also, since we assume that **K** is bounded, symmetric, and uniformly positive definite, we denote by $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ its extreme eigenvalues:

$$\forall \boldsymbol{x} \in \boldsymbol{\Omega}_1, \forall \boldsymbol{\chi} \in \mathbb{R}^3, \, \lambda_{\min} |\boldsymbol{\chi}|^2 \leq \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{\chi} \cdot \boldsymbol{\chi} \leq \lambda_{\max} |\boldsymbol{\chi}|^2, \tag{16}$$

where $|\cdot|$ denotes the Euclidean vector norm.

2.2 Challenges

This coupled problem is challenging, even without the nonlinear convection term. The first difficulty lies in the meaning to be given to the interface conditions involving the traction vector Tn when the interface is not a smooth curve. The next difficulty arises from the nonlinear term: the interface conditions do not eliminate it

from the energy balance. Finally, the numerical implementation of its discretization is problematic because the system is usually large and has different time scales and space scales in each subdomain, whence the necessity of decoupling algorithms.

2.3 Meaning of the Interface Conditions

Consider the following spaces for the data: $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, and assume for the moment that a solution $(\boldsymbol{u}, p_1, p_2)$ exists. It follows easily by inspection that a reasonable choice of spaces for the solution is $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_1)$, $p_1 \in L^2(\Omega_1)$, and $p_2 \in H^1(\Omega_2)$.

Let us start with the simpler situation of the Darcy equations in Ω_2 . The facts that p_2 belongs to $H^1(\Omega_2)$ and **K** is uniformly bounded imply that $\mathbf{K}\nabla p_2$ belongs to $L^2(\Omega_2)$. Then the fact that f_2 belongs to $L^2(\Omega_2)$ and equation (12) imply that $\mathbf{K}\nabla p_2$ is in $\mathbf{H}(\operatorname{div}; \Omega_2)$, where for any domain Ω ,

$$\boldsymbol{H}(\operatorname{div};\boldsymbol{\Omega}) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\boldsymbol{\Omega}); \operatorname{div} \boldsymbol{v} \in L^2(\boldsymbol{\Omega})\}.$$

Therefore $\mathbf{K}\nabla p_2 \cdot \mathbf{n}$ is in $H^{-1/2}(\partial \Omega_2)$, the normal trace space of $\mathbf{H}(\operatorname{div}; \Omega_2)$; it is the dual space of $H^{1/2}(\partial \Omega_2)$, which in turn is the trace space of $H^1(\Omega_2)$; see [31]. In particular, $(\mathbf{K}\nabla p_2) \cdot \mathbf{n}_{12}$ is in $(H_{00}^{1/2}(\Gamma_{12}))'$, the dual space of $H_{00}^{1/2}(\Gamma_{12})$, where $H_{00}^{1/2}(\Gamma_{12})$ is the trace space of functions v in $H^1(\Omega_2)$ that vanish on portions of Γ_2 adjacent to Γ_{12} ; see [46]. Hence $(\mathbf{K}\nabla p_2) \cdot \mathbf{n}_{12}$ is well defined in a weak space. On the other hand, since \mathbf{u} is in $\mathbf{H}^1(\Omega_1)$, then its trace is in $\mathbf{H}^{1/2}(\Gamma_{12})$. Thus Sobolev's imbeddings imply that $\mathbf{u} \cdot \mathbf{n}_{12}$ is in $L^4(\Gamma_{12})$; see [1]. Therefore the equation on Γ_{12}

$$-\mathbf{K}\nabla p_2 \cdot \mathbf{n}_{12} = \mathbf{u} \cdot \mathbf{n}_{12}$$

makes sense and implies that $-\mathbf{K}\nabla p_2 \cdot \mathbf{n}_{12}$ belongs in fact to $L^4(\Gamma_{12})$.

Now we turn to the Navier–Stokes equations in Ω_1 . Since \boldsymbol{u} belongs to $\boldsymbol{H}^1(\Omega_1)$ and p_1 to $L^2(\Omega_1)$, then $\boldsymbol{T}(\boldsymbol{u}, p_1)$ is in $\boldsymbol{L}^2(\Omega_1)$ and $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ belongs to $\boldsymbol{L}^{3/2}(\Omega_1)$. Therefore it follows from (11) that \boldsymbol{T} is in $\boldsymbol{H}^{2,3/2}(\operatorname{div};\Omega_1)$, where

$$\boldsymbol{H}^{2,3/2}(\operatorname{div};\boldsymbol{\Omega}_1) = \{\boldsymbol{\nu} \in \boldsymbol{L}^2(\boldsymbol{\Omega}_1); \operatorname{div} \boldsymbol{\nu} \in L^{3/2}(\boldsymbol{\Omega}_1)\}.$$

As the smooth functions are dense in $H^{2,3/2}(\text{div};\Omega_1)$, then the following Green's formula holds:

$$\forall \boldsymbol{\varphi} \in H^1(\boldsymbol{\Omega}_1), (\operatorname{div} \boldsymbol{\nu}, \boldsymbol{\varphi}) + (\boldsymbol{\nu}, \nabla \boldsymbol{\varphi}) = \langle \boldsymbol{\nu} \cdot \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\partial \boldsymbol{\Omega}_1}.$$

This implies that Tn_{12} is well defined as an element of $(H_{00}^{1/2}(\Gamma_{12}))'$, but if Γ_{12} has corners, the normal and tangent vectors have jumps, and the pairings $\langle Tn_{12}, n_{12} \rangle$ and

 $\langle Tn_{12}, t_{12}^j \rangle$ are not defined. This difficulty can be bypassed by prescribing the last two conditions in (13) simultaneously as a single condition, instead of separately; see [32]. Indeed, set

$$\boldsymbol{g} = p_2 \boldsymbol{n}_{12} + \sum_{j=1}^2 \frac{1}{G^j} \left(\boldsymbol{u} \cdot \boldsymbol{t}_{12}^j \right) \boldsymbol{t}_{12}^j;$$

it is easy to check that **g** belongs to $L^4(\Gamma_{12})$. Since Tn_{12} is well defined, albeit in a weak space, we can prescribe on Γ_{12} :

$$T n_{12} = g$$
.

This condition makes sense and implies that Tn_{12} is in fact in $L^4(\Gamma_{12})$. Then this extra regularity allows to define the above pairings and we recover the last two conditions in (13).

2.4 Variational Formulations

The boundary conditions (14) suggest that we take \boldsymbol{u} and the velocity test functions in

$$\boldsymbol{H}_{\Gamma_1}^1(\boldsymbol{\Omega}_1) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\boldsymbol{\Omega}_1) \, ; \, \boldsymbol{v}|_{\Gamma_1} = \boldsymbol{0} \right\},\,$$

and p_2 and the pressure test functions in

$$H^1_{\Gamma_{2D}}(\Omega_2) = \left\{ q \in H^1(\Omega_2); q|_{\Gamma_{2D}} = 0
ight\}.$$

In these spaces, the system (11)–(14) has the equivalent variational formulation: Find $\boldsymbol{u} \in \boldsymbol{H}_{\Gamma_1}^1(\Omega_1)$, $p_1 \in L^2(\Omega_1)$, and $p_2 \in H_{\Gamma_{2D}}^1(\Omega_2)$, satisfying for all $\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_1}^1(\Omega_1)$, $q_1 \in L^2(\Omega_1)$, and $q_2 \in H_{\Gamma_{2D}}^1(\Omega_2)$:

$$2\mu \left(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v})\right)_{\Omega_{1}} + \left(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}\right)_{\Omega_{1}} - \left(p_{1}, \operatorname{div} \boldsymbol{v}\right)_{\Omega_{1}} + \left(\boldsymbol{K} \nabla p_{2}, \nabla q_{2}\right)_{\Omega_{2}} + \left(p_{2}, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} - \left(q_{2}, \boldsymbol{u} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} + \sum_{j=1}^{2} \left(\frac{1}{G^{j}}\boldsymbol{u} \cdot \boldsymbol{t}_{12}^{j}, \boldsymbol{v} \cdot \boldsymbol{t}_{12}^{j}\right)_{\Gamma_{12}} - \left(\operatorname{div} \boldsymbol{u}, q_{1}\right)_{\Omega_{1}} = \left(\boldsymbol{f}_{1}, \boldsymbol{v}\right)_{\Omega_{1}} + \left(f_{2}, q_{2}\right)_{\Omega_{2}}.$$
(17)

As usual, the pressure p_1 can be eliminated by restricting the test functions to

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_1}^1(\boldsymbol{\Omega}_1); \forall q_1 \in L^2(\boldsymbol{\Omega}_1), (\operatorname{div} \boldsymbol{u}, q_1)_{\boldsymbol{\Omega}_1} = 0\},\$$

and we obtain a reduced equivalent problem: Find $\boldsymbol{u} \in \boldsymbol{V}$ and $p_2 \in H^1_{\Gamma_{2D}}(\Omega_2)$, satisfying for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q_2 \in H^1_{\Gamma_{2D}}(\Omega_2)$:

$$2\mu \left(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v})\right)_{\Omega_{1}} + \left(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}\right)_{\Omega_{1}} + \left(\boldsymbol{K} \nabla p_{2}, \nabla q_{2}\right)_{\Omega_{2}} + \left(p_{2}, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} - \left(q_{2}, \boldsymbol{u} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} + \sum_{j=1}^{2} \left(\frac{1}{G^{j}}\boldsymbol{u} \cdot \boldsymbol{t}_{12}^{j}, \boldsymbol{v} \cdot \boldsymbol{t}_{12}^{j}\right)_{\Gamma_{12}} = \left(\boldsymbol{f}_{1}, \boldsymbol{v}\right)_{\Omega_{1}} + \left(f_{2}, q_{2}\right)_{\Omega_{2}}.$$
(18)

Equivalence follows easily from the inf-sup condition [31]: There exists $\beta > 0$ such that

$$\forall q_1 \in L^2(\Omega_1), \sup_{\boldsymbol{\nu} \in \boldsymbol{H}_{\Gamma_1}^1(\Omega_1)} \frac{(\operatorname{div} \boldsymbol{\nu}, q_1)_{\Omega_1}}{|\boldsymbol{\nu}|_{\boldsymbol{H}^1(\Omega_1)}} \ge \beta \|q_1\|_{L^2(\Omega_1)}.$$
 (19)

2.5 "Energy" Equality and Analysis

The influence of the interface condition on the nonlinear term is clearly illustrated by a straightforward "energy" analysis of problem (18). Assume (18) has a solution (u, p_2) and take v = u, $q_2 = p_2$. Then we readily obtain

$$2\mu \|\boldsymbol{D}(\boldsymbol{u})\|_{\boldsymbol{L}^{2}(\Omega_{1})}^{2} + \left\|\boldsymbol{K}^{1/2}\nabla p_{2}\right\|_{\boldsymbol{L}^{2}(\Omega_{2})}^{2} + \sum_{j=1}^{2} \left\|\left(\frac{1}{G^{j}}\right)^{1/2}\boldsymbol{u}\cdot\boldsymbol{t}_{12}^{j}\right\|_{\boldsymbol{L}^{2}(\Gamma_{12})}^{2} + \frac{1}{2}\int_{\Gamma_{12}}(\boldsymbol{u}\cdot\boldsymbol{n}_{12})|\boldsymbol{u}|^{2} = (\boldsymbol{f}_{1},\boldsymbol{u})_{\Omega_{1}} + (f_{2},p_{2})_{\Omega_{2}}.$$
(20)

The integrand $(\boldsymbol{u} \cdot \boldsymbol{n}_{12})|\boldsymbol{u}|^2$ on Γ_{12} has no definite sign because div $\boldsymbol{u} = 0$ in Ω_1 and $\boldsymbol{u} = \boldsymbol{0}$ on Γ_1 imply that $\boldsymbol{u} \cdot \boldsymbol{n}_{12}$ changes sign on Γ_{12} . But even in the presence of other boundary conditions, one can expect exchanges of fluid at the interface. Therefore in (20) we need to control this integral on the interface.

There are different approaches for treating this integral and establishing existence of solutions. Considering that the difficulty is located on the interface, Badea et al. in [8] reduce problem (18) to a nonlinear interface problem via a nonlinear Steklov–Poincaré operator. Their main unknown is $\lambda = \mathbf{u} \cdot \mathbf{n}_{12}$ on Γ_{12} , and they require an extension operator

$$E: \lambda \in H^{1/2}_{00}(\Gamma_{12}) \mapsto \boldsymbol{v} \in \boldsymbol{H}^1(\boldsymbol{\Omega}_1) \quad \text{satisfying} \quad \boldsymbol{v} \cdot \boldsymbol{n}_{12} = \lambda.$$

Unfortunately, this extension is impossible as soon as Γ_{12} has corners, because in this case \mathbf{n}_{12} is not smooth enough to guarantee that $\mathbf{v} \cdot \mathbf{n}_{12}$ belongs to $H^{1/2}(\Gamma_{12})$. In other words, their approach does not extend to a rough boundary.

This limitation can be avoided by a direct argument (see Girault and Rivière [32]) based on a Galerkin discretization of (18), a priori estimates for restricted data, and Brouwer's fixed point theorem. More precisely, we choose and truncate a smooth basis of $\mathbf{V} \times L^2(\Omega_2)$, say $\mathbf{W}_m = \text{Vect}\{(\Phi_i, \varphi_i)_{1 \le i \le m}\}$, and we want to find $(\mathbf{u}_m, p_m) \in \mathbf{W}_m$ solution of

$$2\mu \left(\boldsymbol{D}(\boldsymbol{u}_{m}), \boldsymbol{D}(\boldsymbol{\Phi}_{k})\right)_{\Omega_{1}} + \left(\boldsymbol{u}_{m} \cdot \nabla \boldsymbol{u}_{m}, \boldsymbol{\Phi}_{k}\right)_{\Omega_{1}} + \left(\boldsymbol{K} \nabla p_{m}, \nabla \varphi_{k}\right)_{\Omega_{2}} \\ + \left(p_{m}, \boldsymbol{\Phi}_{k} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} - \left(\varphi_{k}, \boldsymbol{u}_{m} \cdot \boldsymbol{n}_{12}\right)_{\Gamma_{12}} + \sum_{j=1}^{2} \left(\frac{1}{G^{j}}\boldsymbol{u}_{m} \cdot \boldsymbol{t}_{12}^{j}, \boldsymbol{\Phi}_{k} \cdot \boldsymbol{t}_{12}^{j}\right)_{\Gamma_{12}} \\ = \left(\boldsymbol{f}_{1}, \boldsymbol{\Phi}_{k}\right)_{\Omega_{1}} + \left(f_{2}, \varphi_{k}\right)_{\Omega_{2}}, \quad 1 \leq k \leq m.$$

$$(21)$$

Clearly, any solution of (21) satisfies the energy equality (20). Hence the assumptions of Brouwer's fixed point theorem cannot be checked without restricting the data. With that in mind, it can be readily shown that there exists a constant \mathscr{A} of the form

$$\mathscr{A} = C_1 \|\boldsymbol{f}_1\|_{\boldsymbol{L}^2(\boldsymbol{\Omega}_1)} + \sqrt{\frac{\mu}{\lambda_{\min}}} C_2 \|f_2\|_{L^2(\boldsymbol{\Omega}_2)}$$

with C_1 and C_2 depending only on the geometry of the domain, such that if

$$\mu^2 > \mathscr{CA},\tag{22}$$

where also \mathscr{C} only depends on the geometry of the domain, then (21) has at least one solution \boldsymbol{u}_m, p_m satisfying

$$\mu \left\| \boldsymbol{D}(\boldsymbol{u}_m) \right\|_{\boldsymbol{L}^2(\Omega_1)}^2 + \left\| \boldsymbol{K}^{1/2} \nabla p_m \right\|_{\boldsymbol{L}^2(\Omega_2)}^2 \le \frac{\mathscr{A}^2}{\mu}.$$
(23)

In other words, there exist solutions of (21) for large viscosity or small forces, or both. Furthermore, (22) and (23) imply that

$$\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{\boldsymbol{L}^2(\Omega_1)} \leq \frac{\mathscr{A}}{\mu} < \frac{\mu}{\mathscr{C}} \quad \text{and} \quad \|\boldsymbol{K}^{1/2} \nabla p_m\|_{\boldsymbol{L}^2(\Omega_2)} \leq \frac{\mathscr{A}}{\sqrt{\mu}} < \frac{\mu^{3/2}}{\mathscr{C}}.$$

By a standard argument, these bounds are sufficient to pass to the limit in (21) as *m* tends to infinity. Therefore, provided (22) holds, (18) has at least one solution, and this solution satisfies

$$\|\boldsymbol{D}(\boldsymbol{u})\|_{\boldsymbol{L}^{2}(\Omega_{1})} < \frac{\mu}{\mathscr{C}} \quad \text{and} \quad \|\boldsymbol{K}^{1/2} \nabla p\|_{\boldsymbol{L}^{2}(\Omega_{2})} < \frac{\mu^{3/2}}{\mathscr{C}}.$$
 (24)

Finally, it is easy to prove that (18) has no other solution satisfying (24). Existence of p_1 such that \boldsymbol{u}, p_1, p_2 solves (17) follows from the equivalence of these two formulations.

Remark 2.1. In the case of coupled Stokes and Darcy equations with the same interface conditions (6), (7), and (10), the argument is much simpler. Existence and uniqueness of $(\boldsymbol{u}, p_1, p_2)$ satisfying (17) without the nonlinear term are obtained unconditionally.

3 Discretization

There are several numerical methods that approximate the solution of the Stokes version of (11)–(14), either in divergence form or not. Most have straightforward extensions to the Navier–Stokes equations, although these extensions have not always been proposed. We describe some of them in this section.

3.1 A Discontinuous Galerkin Method

This method has been studied mostly by Girault, Rivière, and Yotov in [32, 49, 51]. Since the analysis presented above applies to a rough interface, we can assume that both Ω_1 and Ω_2 are polygons or polyhedra. This is a major simplification because performing the numerical analysis of problem (11)–(14) in a region with a curved interface raises very technical issues, unless the interface is flat, which is a strong limitation on the geometry.

Let \mathscr{E}_i^h be a regular family (in the sense of Ciarlet [18]) of triangulations of Ω_i made of simplicial elements, i.e., there exists a constant $\gamma > 0$ independent of *h*, such that

$$\forall E \in \mathscr{E}_i^h, \frac{h_E}{\rho_E} = \gamma_E \leq \gamma,$$

where h_E is the diameter of E, ρ_E is the diameter of the ball inscribed in E, and h is the maximum of h_E . Hexahedral elements can also be used, but the nonlinear transformation from the reference cell makes the analysis more technical. As we work with totally discontinuous finite elements, we accept hanging nodes, but for the sake of simplicity, we assume that the triangulations are conforming, and in particular, we assume that the triangulations \mathscr{E}_i^h match on the interface. However, this restriction can be easily relaxed.

The method presented here uses completely discontinuous symmetric interior penalty (SIPG) or nonsymmetric interior penalty (NIPG) everywhere for the elliptic terms; see [41, 50, 52]. This permits to prescribe weakly the essential boundary